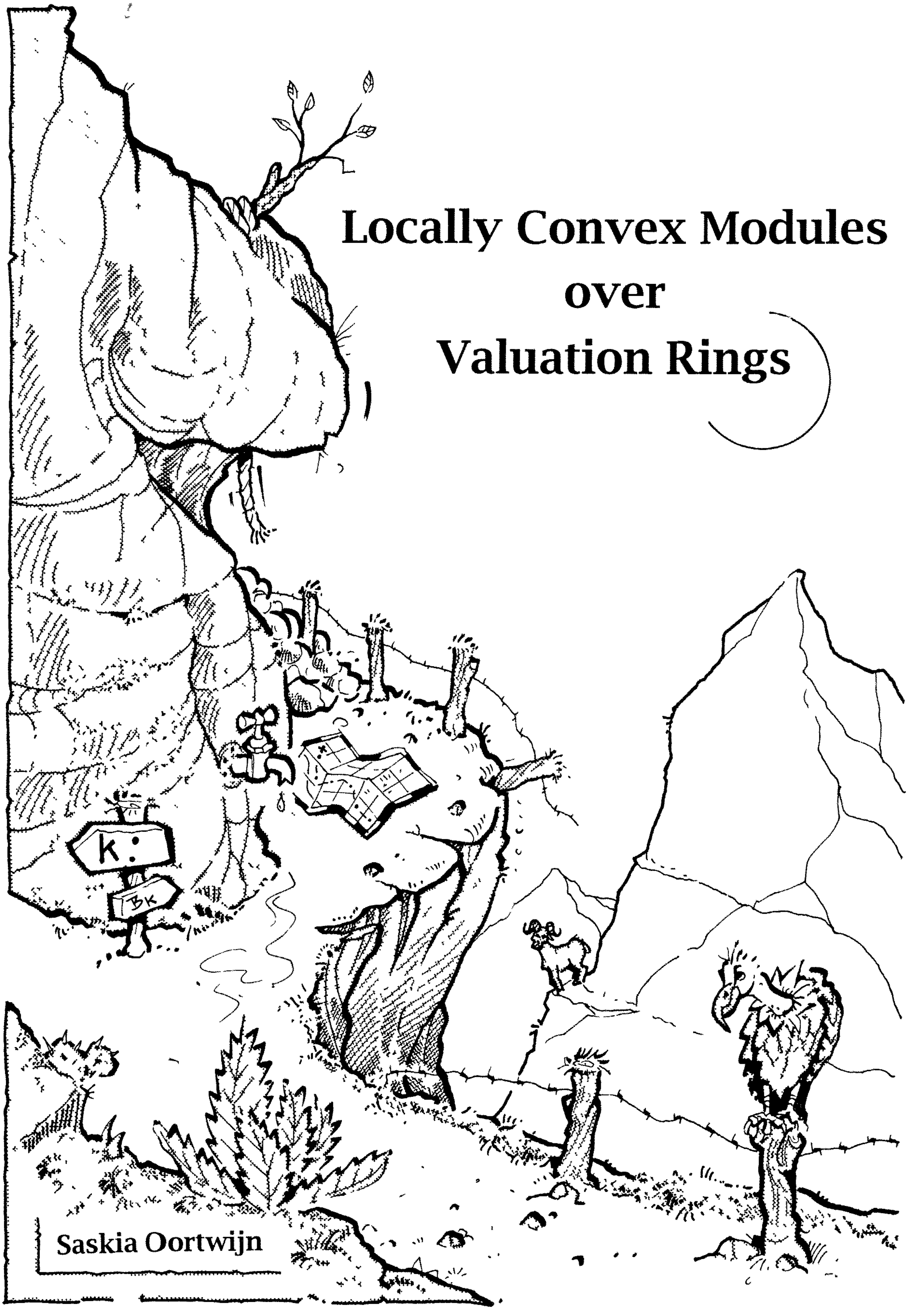
Locally Convex Modules over Valuation Rings



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Locally Convex Modules over Valuation Rings

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proefschrift

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voor Leon

Voorwoord

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Chapter 1

Introduction

In Non-Archimedean Functional Analysis (see [20] for a background account) one is concerned with complete valued base fields K other than $\mathbb{R}$ or $\mathbb{C}$, such as the field of p-adic numbers. The valuation $|\ |$ on K satisfies the *strong triangle inequality*

$$|\lambda + \mu| \leq \max(|\lambda|, |\mu|) \quad (\lambda, \mu \in K)$$

which has far-reaching consequences, for example it follows that

$$B_K := \{\lambda \in K \mid |\lambda| \leq 1\}$$

is a ring; a so-called valuation ring. Usually seminorms p on a K-vector space E are also required to satisfy the strong triangle inequality

$$p(x + y) \leq \max(p(x), p(y)) \quad (x, y \in E).$$

Monna [15] defined a subset C of a K-vector space E to be *convex* if $x_1, x_2, x_3 \in C$, $\lambda_1, \lambda_2, \lambda_3 \in B_K$, $\sum \lambda_i = 1$ implies $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \in C$. To understand why this definition is suitable observe that (i) for each seminorm p all p-balls (that is a set of the form $\{x \in E \mid p(x - a) < \varepsilon\}$ for some $a \in E$, $\varepsilon > 0$) are convex and that (ii) each convex subset is the intersection of p-balls, where p runs through some collection of seminorms. (See Proposition 4.2 in [25].) It was van Tiel [31] who introduced the notion of a locally convex vector space over K as a topological K-vector space having a base of zero neighborhoods consisting of convex sets. Thus, in Non-Archimedean Functional Analysis the above convex sets take over the role played by the convex sets in real or complex analysis.

If a convex subset C of a K-vector space A contains 0 then it is absolutely convex in the sense that $x_1, x_2 \in C$, $\lambda_1, \lambda_2 \in B_K$ implies $\lambda_1 x_1 + \lambda_2 x_2 \in C$. Stated otherwise — and this is a crucial observation for this thesis — C is a B_K-submodule of E ! And non-empty convex sets are nothing else but additive cosets of absolutely convex sets.

Thus, our convex sets carry a natural algebraic structure, richer than their real counterparts. For example, we may define quotients of convex sets: if

$B \subset A$ are absolutely convex then A/B has the structure of a B_K-module (although it may not be absolutely convex i.e. not isomorphic to a B_K-submodule of some K-vector space).

We did not delve deeply into the algebraic theory of B_K-modules; there exists extensive literature, see for instance [11]. In Chapter 2 we have collected the material we needed. The purpose of this thesis, however, is to study *normed* and *locally convex* B_K-modules thereby proposing a new tool in Non-Archimedean Functional Analysis. Hence, for example, rather than considering A/B algebraically we assume that A, B are closed in a normed space and introduce the quotient norm on A/B by the formula

$$\|x + B\| = \inf_{b \in B} \|x - b\| \quad (x \in A).$$

This norm satisfies the strong triangle inequality and

$$\|\lambda x + B\| \leq |\lambda| \|x + B\| \quad (\lambda \in B_K, \ x \in A)$$

but equality does not hold in general. Thus, the requirements for a norm will be relaxed compared to the standard ones (see Definition 2.8 in [31]).

In Chapter 3 and 4 we develop some general theory on locally convex B_K-modules. Apart from some very basic and natural questions we treated material needed for the main Chapter 5 only. Here we characterize the (local) compactoids among all locally convex B_K-modules. It grew out of an informal note by S. Caenepeel [8] on behalf of the p-adic Seminar of the VUB in Brussels. We feel that the existing theory of absolutely convex (local) compactoids of [19], [21] and [26] is better understood in the framework of general locally convex B_K-modules. In fact, the thesis was actuated by the following problem.

Let E, F be K-Banach spaces, let $T \in \mathcal{L}(E, F)$ and let $C \subset E$ be a absolutely convex complete compactoid. How open is the map $T : C \longrightarrow TC$?

The answer, given in [27], can be made more accurate by using the plus and minus topology. See Theorem 5.4.10 in Chapter 5.
For more about the content of this thesis we refer to the Summary on page 187.

The concept of a normed B_K-module also appeared in [5], where for a purpose different from ours, some basic facts were proved.
In [4], [3] and [2] one meets a related but different approach: an absolutely convex closed subset A of a K-Banach space is studied by looking at the quotient $A/\overline{A^i}$, where $A^i = \bigcup_{|\lambda|<1} \lambda A$, not viewed as a B_K-module but considered as a Banach space over the residue class field of K.

Chapter 2

B_K-modules

2.1 Elementary Algebra

In this section we will give some definitions, propositions and theorems from elementary algebra concerning B_K-modules. We also introduce some notation. Furthermore, we provide some examples to illustrate the difference between B_K-modules and (absolutely convex subsets of) K-vector spaces. The whole theory in this chapter, except that of *edged completeness* in Section 2.4, is for the use of the rest of the thesis.
The proofs of the propositions and theorems given in this section can also be found in books about modules (for instance, [12], [1]). Yet they are given here to make the reader, who is used to work with K-vector spaces, familiar with some basic technics in (locally convex) B_K-module theory.

First we recall some notation. $(K, |\ |)$ or, shortly, K denotes a field with a non-archimedean valuation $|\ |$. Our fields are always commutative. For a prime number p, $(\mathbb{Q}_p, |\ |_p)$ or, shortly, $\mathbb{Q}_p$ denotes the field of the p-adic numbers and $(\mathbb{C}_p, |\ |_p)$ or, shortly, $\mathbb{C}_p$ denotes the field of the p-adic complex numbers. (See [22] pages 10 and 45.)
Recall from [20] that a non-archimedean valuation $|\ |$ on a field K is a map $K \to [0, \infty)$ such that
1) $|\lambda| = 0 \iff \lambda = 0 \quad (\lambda \in K)$,
2) $|\lambda\mu| = |\lambda||\mu| \quad (\lambda, \mu \in K)$,
3) $|\lambda + \mu| \leq \max(|\lambda|, |\mu|) \quad (\lambda, \mu \in K)$.

For a subset V of K we denote the set $\{|\lambda| \mid \lambda \in V\}$ by $|V|$.
The valuation $|\ |$ on K is called *dense* if $|K|$ is dense in $[0, \infty)$, $|\ |$ is called *discrete* if $|K| \setminus \{0\}$ is discrete in $(0, \infty)$ and $|\ |$ is called *trivial* if $|K| = \{0, 1\}$.
A *ball* in K is a set of the form $\{\lambda \in K \mid |\lambda - a| \leq r\}$ or $\{\lambda \in K \mid |\lambda - a| < r\}$, where $a \in K$ and $r \in (0, \infty)$. The first ball is denoted $\mathrm{B}(a, r)$; the second one is denoted $\mathrm{B}(a, r^-)$. If $r \notin |K|$ then $\mathrm{B}(a, r) = \mathrm{B}(a, r^-)$.
K is called *spherically complete* if for every collection $(B_n)_{n \in \mathbb{N}}$ of balls in K such that $B_0 \supset B_1 \supset B_2 \supset \ldots$ it holds that $\bigcap_{n \in \mathbb{N}} B_n \neq \emptyset$.

The unit ball of K is denoted B_K, that is to say

$$B_K = \{\lambda \in K \mid |\lambda| \leq 1\}.$$

B_K is a ring; a so-called *valuation ring.* B_K^- denotes the unique maximal ideal of B_K, i.e.

$$B_K^- = \{\lambda \in K \mid |\lambda| < 1\}.$$

The field B_K/B_K^- is called the *residue class field* and is denoted by $\mathbf{k}$.

For elementary theory on non-archimedean valuations we refer to chapter 1 of [20].

We assume elementary algebraic K-vector space theory to be known. We introduce here only some notation. For a K-vector space E we write $\dim E$ for the dimension of E. For a subset X of a K-vector space E the linear span of X is denoted $[X]$.

We end with with some very general notation.
For a subset X of $\mathbb{R}$ we denote by $\operatorname{conv} X$ the convex hull of X; that is to say $\operatorname{conv} X = \{tx + (1-t)y \mid x, y \in X,\ t \in [0,1]\}$.

Let A be a set and let $f : A \to [0,\infty)$ be a map. Then f is called *bounded* if $\sup\{f(x) \mid x \in A\} < \infty$.
For a bounded map $f : A \to [0,\infty)$ we define $\sup f = \sup\{f(x) \mid x \in A\}$.

Definitions and Notation

2.1.1 Definition A B_K*-module* is an abelian group A together with a map $(\lambda, x) \mapsto \lambda x$ from $B_K \times A$ to A, called *scalar multiplication*, satisfying the following properties.

M1. $\lambda(x_1 + x_2) = \lambda x_1 + \lambda x_2 \quad (\lambda \in B_K,\ x_1, x_2 \in A)$

M2. $(\lambda_1 + \lambda_2)x = \lambda_1 x + \lambda_2 x \quad (\lambda_1, \lambda_2 \in B_K,\ x \in A)$

M3. $(\lambda_1 \lambda_2)x = \lambda_1(\lambda_2 x) \quad (\lambda_1, \lambda_2 \in B_K,\ x \in A)$

M4. $1x = x \quad (x \in A)$

A (B_K)*-submodule* of a B_K-module A is a subgroup B of A such that $\lambda x \in B$ for every $\lambda \in B_K$ and every $x \in B$.

Special examples of B_K-modules are B_K itself and K-vector spaces

2.1.2 Definition Let E be a K-vector space. A B_K-submodule of E is called an *absolutely convex set.*

Absolutely convex sets are naturally B_K-modules.

2.1.3 Definition Let A and B be B_K-modules. A *homomorphism* from A to B is a map $\varphi : A \to B$ with the following two properties.

(i) $\varphi(x+y) = \varphi(x) + \varphi(y) \quad (x, y \in A)$.

(ii) $\varphi(\lambda x) = \lambda\varphi(x) \quad (\lambda \in B_K, x \in A)$.

Let $\varphi : A \to B$ be a homomorphism and let X be a subset of A. Then the restriction of φ to X is denoted by $\varphi|X$. For a subset X of A we denote the set $\{\varphi(x) \mid x \in X\}$ by $\varphi(X)$. For a subset Y of B we denote the set $\{x \in A \mid \varphi(x) \in Y\}$ by $\varphi^{-1}(Y)$. The set $\varphi^{-1}(\{0\})$ is denoted $\operatorname{Ker}\varphi$.
The set of all homomorphisms $A \to B$ is denoted $\operatorname{Hom}(A, B)$. It is in an obvious way again a B_K-module.

The following proposition is easy to verify.

2.1.4 Proposition *Let A and B be B_K-modules, let $\varphi : A \to B$ be a homomorphism. Let U be a submodule of A and V a submodule of B. Then $\varphi(U)$ is a submodule of B and $\varphi^{-1}(V)$ is a submodule of A.*

2.1.5 Definition Let A and B be B_K-modules. Then A is called *embeddable in B* if there exists an injective homomorphism from A to B. B is called a *homomorphic image of A* if there exists a surjective homomorphism from A to B. A is called *isomorphic* to B if there exists a bijective homomorphism from A to B. We write $A \sim B$ to express that A is isomorphic to B. A bijective homomorphism from a module A in itself is called an *automorphism*.

2.1.6 Definition Let A be a B_K-module and let X be a subset of A. Like in vector space theory, by $\operatorname{co} X$ we denote the submodule of A generated by X. That is to say

$$\operatorname{co} X = \{\lambda_1 x_1 + \cdots + \lambda_n x_n \mid n \in \mathbb{N}, x_1, \ldots, x_n \in X, \lambda_1, \ldots, \lambda_n \in B_K\}.$$

A is called *finitely generated* if there exists a finite subset X of A such that $A = \operatorname{co} X$. A is called *countably generated* if there exists a countable subset X of A such that $A = \operatorname{co} X$.

In section 4.2 we will prove that a submodule of a countably generated B_K-module is again countably generated. If the valuation on K is discrete then every submodule of a finitely generated module is finitely generated (we will see this in Proposition 2.2.38). If the valuation on K is dense this is in general not true. For example, let $A = B_K$. Then $A = \operatorname{co}\{1\}$ and hence A is finitely generated. Now B_K^- is a submodule of A but B_K^- is not finitely generated.

Now we will introduce some machinery. The following proposition is easy to verify.

2.1.7 Proposition *Let A be a B_K-module and let B be a submodule of A. On the group A/B we define a scalar multiplication by*

$$(\lambda, x + B) \mapsto \lambda x + B \quad (\lambda \in B_K,\ x \in A).$$

Then A/B provided with this scalar multiplication is a B_K-module. The map $\pi : A \to A/B$ defined by $x \mapsto x + B \quad (x \in A)$ is a homomorphism.

Sometimes we call the above map π the *canonical map*.

2.1.8 Definition Let I be an index set and, for every $\iota \in I$, let A_ι be a B_K-module. Then the *(direct) product* of $(A_\iota)_{\iota\in I}$ is the set

$$\prod_{\iota\in I} A_\iota = \{x \mid x : I \to \bigcup_{\iota\in I} A_\iota,\ x(\iota) \in A_\iota \quad (\iota \in I)\}.$$

An element x of $\prod_{\iota\in I} A_\iota$ is also denoted $(x(\iota))_{\iota\in I}$ where $x(\iota) \in A_\iota$ for every $\iota \in I$. The (direct) product is in an obvious way again a B_K-module.
For every $j \in I$ the map $P_j : \prod_{\iota\in I} A_\iota \to A_j$ defined by

$$P_j(x) = x(j) \quad (x \in \prod_{\iota\in I} A_\iota)$$

is a surjective homomorphism and is called the *projection map* from $\prod_{\iota\in I} A_\iota$ on A_j.

The *direct sum* of $(A_\iota)_{\iota\in I}$ is the set

$$\bigoplus_{\iota\in I} A_\iota = \{x \in \prod_{\iota\in I} A_\iota \mid x(\iota) \neq 0 \text{ for only finitely many } \iota \in I\}.$$

The direct sum is again a B_K-module. By restriction, for every $j \in I$, we also have a projection map $P_j : \bigoplus_{\iota\in I} A_\iota \to A_j$.

Let A be a B_K-module and let I be an index set. Let $A_\iota = A$ for every $\iota \in I$. Then the direct product of $(A_\iota)_{\iota\in I}$ is also denoted by A^I. The direct sum of $(A_\iota)_{\iota\in I}$ is also denoted by $A^{(I)}$.

2.1.9 Definition Let A be a B_K-module. Let $(A_\iota)_{\iota\in I}$ be a collection of submodules of A. By $\sum_{\iota\in I} A_\iota$ we denote the submodule

$$\{x_1 + \cdots + x_n \mid n \in \mathbb{N},\ \iota_1, \ldots, \iota_n \in I,\ x_1 \in A_{\iota_1}, \ldots, x_n \in A_{\iota_n}\}.$$

If $A = \sum_{\iota\in I} A_\iota$ and in addition for every $n \in \mathbb{N}$, every $\iota_1, \ldots, \iota_n \in I$ and every $x_1 \in A_{\iota_1}, \ldots, x_n \in A_{\iota_n}$:

$$x_1 + \cdots + x_n = 0 \iff x_1 = \ldots = x_n = 0,$$

then $A \sim \bigoplus_{\iota\in I} A_\iota$ (where $\bigoplus_{\iota\in I} A_\iota$ is defined as in Definition 2.1.8) and we write $A = \bigoplus_{\iota\in I} A_\iota$.

2.1.10 Definition An element x of a B_K-module A is called a *torsion element* if there exists a $\lambda \in B_K \setminus \{0\}$ such that $\lambda x = 0$. A B_K-module A is called a *torsion module* if every element of A is a torsion element. A is called *torsion free* if 0 is the only torsion element in A.
A is called *divisible* if for every $x \in A$ and every $\lambda \in B_K \setminus \{0\}$ there exists a $y \in A$ such that $\lambda y = x$.

2.1.11 Definition Let A be a B_K-module. Then the *torsion part* of A is the set $\{x \in A \mid x \text{ is a torsion element}\}$. It is denoted A_t.

2.1.12 Proposition *Let A be a B_K-module. Then A_t is a torsion submodule of A and A/A_t is torsion free.*

Proof: It is easy to check that A_t is a torsion submodule of A. We prove that A/A_t is torsion free. Let $x \in A/A_t$ and $\lambda \in B_K$ such that $\lambda x = 0$. Suppose $\lambda \neq 0$. We prove that $x = 0$. To this end let $u \in A$ be such that $x = u + A_t$. Then $\lambda u + A_t = \lambda x = 0$ and hence $\lambda u \in A_t$. Hence, there exists a $\mu \in B_K \setminus \{0\}$ such that $\mu(\lambda u) = 0$. Then $\mu\lambda \in B_K \setminus \{0\}$ and $(\mu\lambda)u = 0$. Hence, $u \in A_t$ and thus $x = u + A_t = 0$. □

2.1.13 Examples Let the valuation on K be non-trivial. The following is easily verified.

1. $A = K/B_K$ is a divisible torsion B_K-module, $A_t = A$, $A/A_t = \{0\}$.
2. Let e_1, e_2 be the canonical base for K^2. Then $A = ([e_1] + \operatorname{co}\{e_2\})/\operatorname{co}\{e_1\}$ is neither torsion free nor a torsion B_K-module. It is also not divisible. Furthermore, $A_t \sim [e_1]/\operatorname{co}\{e_1\}$ and $A/A_t \sim \operatorname{co}\{e_2\}$.

If the valuation on K is trivial then each B_K-module is also a K-vector space. In general, a K-vector space E is a B_K-module that is divisible and torsion free. We also have the following converse that we will prove at the end of this section, after Lemma 2.1.34

2.1.14 Theorem *Let A be a B_K-module that is divisible and torsion free. Then the scalar multiplication $B_K \times A \to A$ can be extended uniquely to a scalar multiplication $K \times A \to A$ making A into a K-vector space.*

On Scalar Multiplication

On B_K-modules the scalar multiplication works differently from that on K-vector spaces, on which the scalar multiplication by a scalar $\neq 0$ is a bijective linear map. For B_K-modules we have the following proposition.

2.1.15 Proposition *Let A be a B_K-module and $\lambda \in B_K \setminus \{0\}$. Then the map $M_\lambda : A \to A$ defined by $M_\lambda(x) = \lambda x \quad (x \in A)$ is a homomorphism. Furthermore, M_λ is injective if and only if A is torsion free and M_λ is surjective if and only if A is divisible. If $|\lambda| = 1$ the map M_λ is an automorphism.*

2.1.16 Definition Let A be a B_K-module and X a subset of A. Let $\lambda \in K$.
If $|\lambda| \leq 1$ we define $\lambda X = \{\lambda x \mid x \in X\}$.
If $|\lambda| > 1$ we define $\lambda X = \{x \in A \mid \lambda^{-1} x \in X\}$.

It is easily verified that $\{\lambda x \mid x \in X\} = \{x \in A \mid \lambda^{-1} x \in X\}$ for all $\lambda \in B_K$ with $|\lambda| = 1$.

2.1.17 Proposition *Let A be a B_K-module and let B be a submodule of A. Let $\lambda, \mu \in K$ such that $|\lambda| \geq |\mu|$. Then λB and μB are submodules of A and $\mu B \subset \lambda B$. In particular, $|\lambda| = |\mu|$ implies $\lambda B = \mu B$.*

Proof: First we prove that λB and μB are submodules of A.
If $|\lambda| \leq 1$ then $\lambda B = M_\lambda(B)$. Combining the Propositions 2.1.15 and 2.1.4 we obtain that λB is a submodule of A. If $|\lambda| > 1$ then $\lambda B = M_{\lambda^{-1}}^{-1}(B)$ which is again a submodule of A. In the same way we obtain that μB is a submodule of A.
Now suppose $|\mu| \leq 1$. Let $x \in \mu B$. Then there exists a $u \in B$ such that $x = \mu u$. If $|\lambda| \leq 1$ then $(\lambda^{-1}\mu)u \in B$ and $\lambda(\lambda^{-1}\mu)u = \mu u = x$. Hence, $x \in \lambda B$. If $|\lambda| > 1$ then $\mu B \subset B \subset \lambda B$ as is easy to verify.
Suppose $|\mu| > 1$. Then also $|\lambda| > 1$. Let $x \in \mu B$. Then $\mu^{-1}x \in B$. Hence, $\lambda^{-1}x = (\lambda^{-1}\mu)\mu^{-1}x \in B$ and thus $x \in \lambda B$. □

The next proposition is about compositions of scalar multiplication.

2.1.18 Proposition *Let A be a B_K-module and let X be a subset of A. Let $\lambda, \mu \in K$.*

1. *If $|\lambda|, |\mu| \leq 1$ then $\lambda(\mu X) = (\lambda\mu)X$.*

2. *If $|\lambda|, |\mu| > 1$ then $\lambda(\mu X) = (\lambda\mu)X$.*

3. *If $|\lambda| \leq 1$ and $|\mu| > 1$ then $\lambda(\mu X) \subset (\lambda\mu)X$.*
   If A is divisible then $\lambda(\mu X) = (\lambda\mu)X$.

4. *If $|\lambda| > 1$ and $|\mu| \leq 1$ then $\lambda(\mu X) \supset (\lambda\mu)X$.*
   If A is torsion free then $\lambda(\mu X) = (\lambda\mu)X$.

Proof: The proofs of 1. and 2. are straightforward. We prove 3. The proof of 4. is similar. We first prove that $\lambda(\mu X) \subset (\lambda\mu)X$. Let $x \in \lambda(\mu X)$. Then there exists a $y \in \mu X$ such that $x = \lambda y$. Then $\mu^{-1}y \in X$. Suppose $|\lambda\mu| \leq 1$. Then $x = \lambda y = (\lambda\mu\mu^{-1})y = (\lambda\mu)(\mu^{-1}y) \in (\lambda\mu)X$. Suppose $|\lambda\mu| > 1$. Then $(\lambda\mu)^{-1}x = (\mu^{-1}\lambda^{-1})x = (\mu^{-1}\lambda^{-1})(\lambda y) = (\mu^{-1}\lambda^{-1}\lambda)y = \mu^{-1}y \in X$ and hence $x \in (\lambda\mu)X$.
Suppose A is divisible. Let $x \in (\lambda\mu)X$. Suppose $|\lambda\mu| \leq 1$. Then there exists a $y \in X$ such that $x = (\lambda\mu)y$. As A is divisible there exists a $z \in A$ such that $\mu^{-1}z = y$. Then $z \in \mu X$ and $\lambda z = (\lambda\mu\mu^{-1})z = (\lambda\mu)(\mu^{-1}z) = (\lambda\mu)y = x$ and hence $x \in \lambda(\mu X)$. Suppose $|\lambda\mu| > 1$. There exists a $z \in A$ such that $\lambda z = x$. Then $z \in \lambda^{-1}(\lambda\mu)X = \mu X$ (according to 2.). Hence, $x \in \lambda(\mu X)$. □

Next we will present an example in which the inclusions in 3. and 4. may be strict.

2.1.19 Example Let p be a prime number and let $K = \mathbb{Q}_p$.
Let $A = B_K/p^2B_K$ and let $X = pB_K/p^2B_K$. Then it is easily seen that $p^{-2}(p^2X) = A$, $p^2(p^{-2}X) = \{0\}$ and $(p^{-2}p^2)X = X$.

The following proposition collects some results on homomorphisms.

2.1.20 Proposition *Let A and B be B_K-modules and $T : A \to B$ a homomorphism.*

1. *Let $\lambda \in K$ and let X be a subset of A. If $|\lambda| \leq 1$ then $T(\lambda X) = \lambda T(X)$.*
   If $|\lambda| > 1$ then $T(\lambda X) \subset \lambda T(X)$.
   If, in addition, T is bijective, then $T(\lambda X) = \lambda T(X)$.
2. *Let $\lambda \in K$ and let Y be a subset of B. If $|\lambda| \geq 1$ then $T^{-1}(\lambda Y) = \lambda T^{-1}(Y)$.*
   If $|\lambda| < 1$ then $\lambda T^{-1}(Y) \subset T^{-1}(\lambda Y)$.
   If, in addition, T is bijective, then $\lambda T^{-1}(Y) = T^{-1}(\lambda Y)$.
3. *Let X_1, X_2 be subsets of A. Then $T(X_1 + X_2) = T(X_1) + T(X_2)$.*
4. *Let Y_1, Y_2 be subsets of B. Then $T^{-1}(Y_1) + T^{-1}(Y_2) \subset T^{-1}(Y_1 + Y_2)$.*
   If, in addition, T is surjective then $T^{-1}(Y_1) + T^{-1}(Y_2) = T^{-1}(Y_1 + Y_2)$.

Proof: As an example we prove 4.
We first prove $T^{-1}(Y_1)+T^{-1}(Y_2) \subset T^{-1}(Y_1+Y_2)$. Let $x \in T^{-1}(Y_1)+T^{-1}(Y_2)$. Then $T(x) \in T(T^{-1}(Y_1) + T^{-1}(Y_2)) = T(T^{-1}(Y_1)) + T(T^{-1}(Y_2)) \subset Y_1 + Y_2$. Hence, $x \in T^{-1}(Y_1 + Y_2)$.
Suppose T is surjective. Let $x \in T^{-1}(Y_1 + Y_2)$. Then $T(x) \in Y_1 + Y_2$. Let $y_1 \in Y_1$ and $y_2 \in Y_2$ such that $T(x) = y_1 + y_2$. Let $u_1, u_2 \in A$ such that $T(u_1) = y_1$ and $T(u_2) = y_2$. Then $u_1 \in T^{-1}(Y_1)$ and $u_2 \in T^{-1}(Y_2)$ and $T(x-(u_1+u_2)) = 0$. Thus, $x-(u_1+u_2) \in \operatorname{Ker} T$. Then $x \in u_1+u_2+\operatorname{Ker} T \subset T^{-1}(Y_1) + T^{-1}(Y_2) + \operatorname{Ker} T = T^{-1}(Y_1) + T^{-1}(Y_2)$. □

2.1.21 Corollary *Let A be a B_K-module and let X and Y be subsets of A. Let $\lambda \in K$.*
If $|\lambda| \leq 1$ then $\lambda(X + Y) = \lambda X + \lambda Y$.
If $|\lambda| > 1$ then $\lambda X+\lambda Y \subset \lambda(X+Y)$. If A is divisible, then $\lambda X+\lambda Y = \lambda(X+Y)$.

Proof: For $|\lambda| \leq 1$ apply 3. of the previous proposition with $B = A$, $T = M_\lambda$, $X_1 = X$ and $X_2 = Y$.
For $|\lambda| > 1$ apply 4. of the previous proposition with $B = A$, $T = M_{\lambda^{-1}}$, $Y_1 = X$ and $Y_2 = Y$. Recall that $M_{\lambda^{-1}}$ is surjective if A is divisible. □

We now give some examples to show that the inclusions in Proposition 2.1.20 and Corollary 2.1.21 may be strict.

2.1.22 Examples We start with two examples for which the inclusions in 1. and 2. from Proposition 2.1.20 are strict, the first example concerning an injective map and the second example concerning a surjective map.
Let p be a prime number and let $K = \mathbb{Q}_p$. Let $A = B_K$ and $B = K$. Let $T : A \to B$ be the inclusion map. Then T is injective. Let $X = B_K$. Then $T(p^{-1}X) = B_K$, while $p^{-1}T(X) = p^{-1}B_K$. This shows the inclusion in 1. may be strict.
Let $Y = p^{-1}B_K$. Then $T^{-1}(pY) = B_K$, whereas $pT^{-1}(Y) = pB_K$. This shows the inclusion in 2. may be strict.
Now let $A = B_K \times K$ and $B = K$. Let $T : A \to B$ be defined by $T((\lambda, \mu)) = \lambda + \mu \quad (\lambda \in B_K, \mu \in K)$. Then T is surjective. Let $X = B_K \times \{0\}$. Then $T(p^{-1}X) = B_K$, while $p^{-1}T(X) = p^{-1}B_K$.

Let $Y = p^{-1}B_K$. Then $T^{-1}(pY) = B_K \times B_K$, whereas $pT^{-1}(Y) = pB_K \times B_K$. From K-vector space theory we already know that the inclusion in 4. may be strict. We give an example in which the inclusion in Corollary 2.1.21 is strict.
Again, let p be a prime number and let $K = \mathbb{Q}_p$. Let e_1, e_2 be the canonical base of K^2. Let $A = \operatorname{co}\{pe_1, e_1 + e_2\}$. Let $X = \operatorname{co}\{pe_1\}$ and $Y = \operatorname{co}\{pe_2\}$. Then $p^{-1}X = \operatorname{co}\{pe_1\}$ and $p^{-1}Y = \operatorname{co}\{pe_2\}$. Hence, $p^{-1}X + p^{-1}Y = \operatorname{co}\{pe_1, pe_2\}$, whereas $p^{-1}(X + Y) = p^{-1}\operatorname{co}\{pe_1, pe_2\} = \operatorname{co}\{pe_1, e_1 + e_2\} = A$.

We now introduce the notion of an absorbing subset of a B_K-module, that we will use frequently further on.

2.1.23 Definition Let A be a B_K-module. A subset U of A is called *absorbing* (*in* A) if for every $x \in A$ there exists a $\lambda \in B_K \setminus \{0\}$ such that $\lambda x \in U$.
Let Y be a subset of A. We say *U absorbs all elements of Y* if for every $x \in Y$ there exists a $\lambda \in B_K \setminus \{0\}$ such that $\lambda x \in U$.

Homomorphisms

2.1.24 Proposition *Let E and F be K-vector spaces. Let $\varphi : E \to F$ be a homomorphism. Then φ is linear.*

Proof: Let $x \in E$ and $\lambda \in K$ with $|\lambda| > 1$. Then $x = \lambda^{-1}(\lambda x)$ and hence $\varphi(x) = \lambda^{-1}\varphi(\lambda x)$. That is to say $\varphi(\lambda x) = \lambda\varphi(x)$. □

The following theorem is the well-known (at least for groups and vector spaces) homomorphy theorem.

2.1.25 Theorem *Let A, B be B_K-modules and let $\varphi : A \to B$ be a homomorphism. Then there exists a bijective homomorphism $\rho : A/\operatorname{Ker}\varphi \to \varphi(A)$ such that the following diagram commutes.*

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \varphi(A) \\ A/\operatorname{Ker}\varphi \arrow[ur, "\rho"'] & \end{tikzcd}$$

Here $\pi : A \to A/\operatorname{Ker}\varphi$ is the canonical map.

2.1.26 Corollary *Let A be a B_K-module and let B, C be submodules of A. Then $C/(C \cap B) \sim (B + C)/B$.*

Proof: The map $\varphi : C \to (B + C)/B$ defined by $\varphi(x) = (0 + x) + B \quad (x \in C)$ is a surjective homomorphism and $\operatorname{Ker}\varphi = C \cap B$. Now apply the above theorem. □

It is not hard to prove the following proposition.

2.1.27 Proposition *Let A be a B_K-module and B and C submodules of A such that $C \subset B$. Then there exists a surjective homomorphism $\varphi : A/C \to A/B$ such that the following diagram commutes.*

$$\begin{tikzcd} & A \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ A/C \arrow[rr, "\varphi"'] & & A/B \end{tikzcd}$$

Here $\pi_1 : A \to A/C$ and $\pi_2 : A \to A/B$ are the canonical maps.

2.1.28 Corollary *Let A, B be B_K-modules and let $\varphi : A \to B$ be a surjective homomorphism. Let C be a submodule of A such that $C \subset \operatorname{Ker} \varphi$. Then there exists a surjective homomorphism $\hat{\varphi} : A/C \to B$.*

2.1.29 Theorem *Let A be a B_K-module and $X \subset A$ such that $A = \operatorname{co} X$. Let B be a B_K-module and let $\varphi : X \to B$ be a map with the following property.*
If $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and $\lambda_1, \dots, \lambda_n \in B_K$ such that $\lambda_1 x_1 + \cdots + \lambda_n x_n = 0$ then $\lambda_1 \varphi(x_1) + \cdots + \lambda_n \varphi(x_n) = 0$.
Then there exists a homomorphism $\tilde{\varphi} : A \to B$ such that $\tilde{\varphi}|X = \varphi$.

Proof: Let $y \in A$. There exist $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and $\lambda_1, \dots, \lambda_n \in B_K$ such that $y = \lambda_1 x_1 + \cdots + \lambda_n x_n$. We define

$$\tilde{\varphi}(y) = \lambda_1 \varphi(x_1) + \cdots + \lambda_n \varphi(x_n).$$

This is a good definition. In fact, suppose also $m \in \mathbb{N}$, $z_1, \dots, z_m \in X$ and $\mu_1, \dots \mu_m \in B_K$ such that $y = \mu_1 z_1 + \cdots \mu_m z_m$. Then

$$\lambda_1 x_1 + \cdots + \lambda_n x_n + (-\mu_1) z_1 + \cdots + (-\mu_m) z_m = 0$$

and hence

$$\lambda_1 \varphi(x_1) + \cdots + \lambda_n \varphi(x_n) + (-\mu_1)\varphi(z_1) + \cdots + (-\mu_m)\varphi(z_m) = 0.$$

This implies that

$$\lambda_1 \varphi(x_1) + \cdots + \lambda_n \varphi(x_n) = \mu_1 \varphi(z_1) + \ldots + \mu_m \varphi(z_m).$$

We prove that $\tilde{\varphi}$ is a homomorphism.
Let $u, v \in A$. Let $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and $\lambda_1, \dots, \lambda_n \in B_K$ be such that $u = \lambda_1 x_1 + \cdots + \lambda_n x_n$. Let $m \in \mathbb{N}$, $y_1, \dots, y_m \in X$ and $\mu_1, \dots, \mu_m \in B_K$ be such that $v = \mu_1 y_1 + \cdots + \mu_m y_m$. Then

$$u + v = \lambda_1 x_1 + \cdots + \lambda_n x_n + \mu_1 y_1 + \cdots + \mu_m y_m$$

and hence

$$\begin{aligned} \tilde{\varphi}(u+v) &= \lambda_1 \varphi(x_1) + \cdots + \lambda_n \varphi(x_n) + \mu_1 \varphi(y_1) + \cdots + \mu_m \varphi(y_m) = \\ &(\lambda_1 \varphi(x_1) + \cdots + \lambda_n \varphi(x_n)) + (\mu_1 \varphi(y_1) + \cdots + \mu_m \varphi(y_m)) = \\ \tilde{\varphi}(u) &+ \tilde{\varphi}(v). \end{aligned}$$

In the same way we obtain that $\tilde{\varphi}(\lambda u) = \lambda \tilde{\varphi}(u)$ for $\lambda \in B_K$ and $u \in A$. □

On the Structure of a Module

2.1.30 Lemma *Let A be a divisible B_K-module and let $a \in A$. Then there exists a homomorphism $\varphi : K \to A$ with $\varphi(1) = a$.*

Proof: If the valuation on K is trivial then $K = B_K$ and we define $\varphi : K \to A$ by $\varphi(\lambda) = \lambda a \quad (\lambda \in K)$.
Suppose that the valuation on K is non-trivial. Let $\nu \in K$ such that $|\nu| > 1$. Then $K = \mathrm{co}\{1, \nu, \nu^2, \dots\}$. Let $y_0 = a$. As A is divisible we obtain that there exists a $y_1 \in A$ such that $\nu^{-1}y_1 = y_0$. There exists a $y_2 \in A$ such that $\nu^{-1}y_2 = y_1$. Continuing this way we find $y_0, y_1, y_2, \dots \in A$ such that $\nu^{-1}y_{n+1} = y_n$ for every $n \in \mathbb{N}$. It is not hard to prove that $\nu^{m-n}y_n = y_m$ for every $n, m \in \mathbb{N}$ with $n \geq m$. We define a map $\varphi : \{1, \nu, \nu^2, \dots\} \to A$ by $\varphi(\nu^n) = y_n \quad (n \in \mathbb{N})$.
Let $n \in \mathbb{N}$ and $\lambda_0, \dots, \lambda_n \in B_K$ be such that $\lambda_0 + \lambda_1\nu + \cdots + \lambda_n\nu^n = 0$. Then $\lambda_0 y_0 + \lambda_1 y_1 + \cdots + \lambda_n y_n = \lambda_0 \nu^{-n} y_n + \lambda_1 \nu^{n-1} y_n + \dots + \lambda_n y_n = \nu^{-n}(\lambda_0 + \lambda_1\nu + \cdots + \lambda_n\nu^n)y_n = \nu^n \cdot 0 \cdot y_n = 0$. By applying Theorem 2.1.29 we obtain that there exists a homomorphism $\tilde{\varphi} : K \to A$ such that $\tilde{\varphi}|\{1, \nu, \nu^2, \dots\} = \varphi$. Then $\tilde{\varphi}(1) = \varphi(1) = y_0 = a$. □

2.1.31 Proposition *Let A be a torsion free B_K-module. Then there exist a K-vector space E and an injective homomorphism $i : A \to E$ with $[i(A)] = E$.*

Proof: We define an equivalence relation $\sim$ on $K \times A$ as follows. Let $\lambda, \mu \in K$ and $x, y \in A$. Then $(\lambda, x) \sim (\mu, y)$ if there exists a $\nu \in B_K \setminus \{0\}$ such that $\nu\lambda, \nu\mu \in B_K$ and $(\nu\lambda)x = (\nu\mu)y$.
Let $E = (K \times A)/\sim$. For $\lambda \in K$ and $x \in A$ we denote the equivalence class of (λ, x) by $(\lambda, x)_\sim$. We define an addition on E as follows. Let $\lambda, \mu \in K$ and $x, y \in A$. Let $\nu \in B_K \setminus \{0\}$ such that $\nu\lambda, \nu\mu \in B_K$. Then

$$(\lambda, x)_\sim + (\mu, y)_\sim := (\nu^{-1}, \nu\lambda x + \nu\mu y)_\sim.$$

It is not hard to check that this addition is well defined and that it makes E into a group. We define a scalar multiplication $K \times E \to E$ by

$$\rho(\lambda, x)_\sim := (\rho\lambda, x)_\sim \quad (\rho \in K,\ (\lambda, x) \in K \times E).$$

It takes a standard verification that this scalar multiplication is well defined and that it makes the group E into a K-vector space.
Now the map $i : A \to E$ defined by $i(x) = (1, x)_\sim$ is obviously a homomorphism. The following shows that i is also injective. Let $x \in A$ be such that $i(x) = 0$. Then $(1, x) \sim (1, 0)$ and hence there exists a $\nu \in B_K \setminus \{0\}$ such that $\nu x = \nu 0 = 0$. As A is torsion free it follows that $x = 0$.
Finally we prove that $[i(A)] = E$. To this end, let $u \in E$. Let $\lambda \in K$ and $x \in A$ be such that $u = (\lambda, x)_\sim$. Then $u = (\lambda, x)_\sim = \lambda(1, x)_\sim \in [i(A)]$. □

2.1.32 Remark The K-vector space E in the above proof is known as the tensor product of K and A over B_K and is usually denoted $K \otimes_{B_K} A$. (See, for instance, [1] pages 24–35.)

2.1.33 Proposition

(i) *Every B_K-module is a homomorphic image of an absolutely convex set.*

(ii) *Every divisible B_K-module is a homomorphic image of a K-vector space.*

(iii) *Every B_K-module is embeddable in a divisible B_K-module.*

(iv) *Every torsion free B_K-module is embeddable in a K-vector space.*

Proof: (i) Let A be a B_K-module. Then $B_K^{(A)}$ is a torsion free B_K-module. Now $K^{(A)}$ is a K-vector space and $B_K^{(A)} \subset K^{(A)}$, hence $B_K^{(A)}$ is absolutely convex. Let $\varphi : B_K^{(A)} \to A$ be defined by

$$\varphi(u) = \sum_{x \in A} u(x) \cdot x \quad (u \in B_K^{(A)}).$$

This is a map since the sum $\sum_{x \in A} u(x) \cdot x$ is finite for every $u \in B_K^{(A)}$. It is not hard to prove that φ is a homomorphism. Furthermore, φ is surjective. For let $y \in A$. Let $u \in B_K^{(A)}$ be defined by

$$u(x) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

Then $\varphi(u) = \sum_{x \in A} u(x) \cdot x = 1 \cdot y = y$.
(ii) Let A be a divisible B_K-module. By Lemma 2.1.30 there exists, for every $a \in A$, a homomorphism $\varphi_a : K \to A$ with $\varphi_a(1) = a$. Now $K^{(A)}$ is a K-vector space and we define a map $\rho : K^{(A)} \to A$ by

$$\rho(u) = \sum_{x \in A} \varphi_x(u(x)) \quad (u \in K^{(A)}).$$

In the same way as in the proof of (i) we obtain that ρ is a well-defined, surjective homomorphism.
(iii) Let A be a B_K-module. From (i) we obtain that there exists an absolutely convex subset B of some K-vector space E and a surjective homomorphism $\varphi : B \to A$. From Theorem 2.1.25 it follows that $A \sim B/\operatorname{Ker}\varphi$. Now $E/\operatorname{Ker}\varphi$ is a divisible B_K-module and $\rho : B \to E/\operatorname{Ker}\varphi$ defined by $\rho(x) = x + \operatorname{Ker}\varphi \quad (x \in B)$ is a homomorphism. Again by using Theorem 2.1.25 we obtain that $B/\operatorname{Ker}\varphi (= B/\operatorname{Ker}\rho)$ is embeddable in $E/\operatorname{Ker}\varphi$ and as $A \sim B/\operatorname{Ker}\varphi$ it follows that also A is embeddable in $E/\operatorname{Ker}\varphi$.
(iv) This is Proposition 2.1.31. □

The following lemma we need for the proof of Theorem 2.1.14.

2.1.34 Lemma *Let E be a K-vector space and let A be a B_K-module. Suppose that there exists a bijective homomorphism $\varphi : E \to A$ (that is to say E and A are isomorphic as B_K-modules). Then the scalar multiplication on A can be extended uniquely to a scalar multiplication $K \times A \to A$, making A into a K-vector space.*

Proof: We define a scalar multiplication $\rho : K \times A \to A$ as follows. Let $\lambda \in K$ and $x \in A$. Then $\rho(\lambda, x) := \varphi(\lambda\varphi^{-1}(x))$. It is not hard to see that this scalar multiplication is an extension of the initial scalar multiplication $B_K \times A \to A$. It takes a standard verification to show that A, provided with this scalar multiplication, is a K-vector space.
Suppose $\rho' : K \times A \to A$ is also a scalar multiplication, which is an extension of the initial scalar multiplication $B_K \times A \to A$ and makes A into a K-vector space. Then for $x \in A$ and $\lambda \in K$ with $|\lambda| > 1$ we have that

$$\lambda^{-1}(\rho(\lambda,x) - \rho'(\lambda,x)) = \lambda^{-1}\rho(\lambda,x) - \lambda^{-1}\rho'(\lambda,x) =$$
$$\rho(\lambda^{-1},\rho(\lambda,x)) - \rho'(\lambda^{-1},\rho'(\lambda,x)) = \rho(1,x) - \rho'(1,x) = x - x = 0.$$

As A is torsion free it follows that $\rho(\lambda,x) = \rho'(\lambda,x)$. □

Proof of Theorem 2.1.14: From Proposition 2.1.33 we obtain that there exist a vector space E and a surjective homomorphism $\varphi : E \to A$. From Theorem 2.1.25 we obtain that there exists a bijective homomorphism from $E/\operatorname{Ker}\varphi$ to A.
Let $x \in [\operatorname{Ker}\varphi]$. Then there exists a $\lambda \in B_K \setminus \{0\}$ such that $\lambda x \in \operatorname{Ker}\varphi$. Then $\lambda\varphi(x) = \varphi(\lambda x) = 0$ and as A is torsion free it follows that $\varphi(x) = 0$ and hence $x \in \operatorname{Ker}\varphi$.
We obtain that $[\operatorname{Ker}\varphi] = \operatorname{Ker}\varphi$ and hence $\operatorname{Ker}\varphi$ is a linear subspace of E. Then $E/\operatorname{Ker}\varphi$ is a K-vector space and with the previous lemma we obtain that A is also a K-vector space. □

2.2 B_K-modules of Finite Rank

Most of the things that are proved in this section are well-known facts in algebra. They can be found in [1], [7], [10], and [11].

Finitely generated B_K-modules

2.2.1 Definition Let A be a B_K-module. We define

$$A^- = \{\lambda x \mid \lambda \in B_K^-,\ x \in A\}.$$

For an absolutely convex set A the set A^- is also denoted A^i. It is not hard to prove that A^- is a submodule of A. Hence, A/A^- is a B_K-module. It is also a $\mathbf{k}$-vector space as we will see in the following proposition.

2.2.2 Proposition *Let A be a B_K-module. Then there exists a unique map $\rho : \mathbf{k} \times A/A^- \to A/A^-$ such that the following diagram commutes.*

$$\begin{tikzcd}
B_K \times A \arrow[r, "\pi' \times \pi"] \arrow[d, "\text{scalar multiplication}"'] & \mathbf{k} \times A/A^- \arrow[d, "\rho"] \\
A \arrow[r, "\pi"'] & A/A^-
\end{tikzcd}$$

(Here, $\pi : A \to A/A^-$ is the canonical map and the map $\pi' \times \pi$ is defined by $(\lambda, x) \mapsto (\lambda + B_K^-, x + A^-)$ $(\lambda \in B_K,\ x \in A)$.)
Furthermore, ρ is a scalar multiplication making A/A^- into a $\mathbf{k}$*-vector space.*

Proof: We define $\rho : \mathbf{k} \times A/A^- \to A/A^-$ as follows.
Let $\nu \in \mathbf{k}$ and $x \in A/A^-$. Let $\lambda \in B_K$ and $u \in A$ such that $\nu = \lambda + B_K^-$ and $x = u + A^-$. Then $\nu x := \lambda u + A^-$. Suppose also $\mu \in B_K$ and $v \in A$ such that $\nu = \mu + B_K^-$ and $x = v + A^-$. Then

$$(\mu v - \lambda u) = \mu(v - u) + (\mu - \lambda)u \in \mu A^- + (\mu - \lambda)A \subset A^- + A^- = A^-$$

and hence $\lambda u + A^- = \mu v + A^-$. Thus, ρ is well defined and it is obvious that the above diagram commutes. It takes a standard verification that ρ is a scalar multiplication making A/A^- into a $\mathbf{k}$-vector space. □

2.2.3 Definition Let A be a finitely generated B_K-module. A collection $e_1, \ldots, e_n \in A$ is called a *minimally generating collection for A* if

1) $A = \mathrm{co}\{e_1, \ldots, e_n\}$

2) if also $y_1, \ldots, y_m \in A$ such that $A = \mathrm{co}\{y_1, \ldots, y_m\}$ then $m \geq n$.

2.2.4 Proposition *Let A be a finitely generated B_K-module, say $A = \mathrm{co}\{g_1, \ldots, g_m\}$. Let $x \in A^-$. Then there exist $\lambda_1, \ldots, \lambda_m \in B_K^-$ such that $x = \sum_{i=1}^m \lambda_i g_i$.*

Proof: There exists a $y \in A$ and a $\lambda \in B_K^-$ such that $x = \lambda y$. There exist $\nu_1, \ldots, \nu_m \in B_K$ such that $y = \nu_1 g_1 + \cdots + \nu_m g_m$. Then $\lambda\nu_i \in B_K^-$ for every $i \in \{1, \ldots, m\}$ and $x = \lambda\nu_1 g_1 + \cdots + \lambda\nu_m g_m$. □

2.2.5 Proposition *Let A be a finitely generated B_K-module. Let $e_1, \ldots, e_n$ be a minimally generating collection for A.*
Let $\lambda_1, \ldots, \lambda_n \in B_K$ such that $\lambda_1 e_1 + \cdots + \lambda_n e_n = 0$. Then $|\lambda_i| < 1$ for every $i \in \{1, \ldots, n\}$.

Proof: Suppose there exists an $i \in \{1, \ldots, n\}$ such that $|\lambda_i| = 1$. By renumeration we may assume that $i = n$. Then $\lambda_n^{-1}\lambda_i \in B_K$ for every $i \in \{1, \ldots, n-1\}$ and $e_n = (-\lambda_n^{-1}\lambda_1)e_1 + \cdots + (-\lambda_n^{-1}\lambda_{n-1})e_{n-1} \in \mathrm{co}\{e_1, \ldots, e_{n-1}\}$. Hence, $\mathrm{co}\{e_1, \ldots, e_n\} = \mathrm{co}\{e_1, \ldots, e_{n-1}\}$. This is in contradiction with the minimality of n. Thus, $|\lambda_i| < 1$ for every $i \in \{1, \ldots, n\}$. □

2.2.6 Proposition *Let A be a finitely generated B_K-module and let $e_1, \ldots, e_n$ be a minimally generating collection for A. Let $x \in A$ and $\lambda_1, \ldots, \lambda_n \in B_K$ such that $x = \sum_{i=1}^n \lambda_i e_i$. Then*

$$x \in A^- \iff |\lambda_i| < 1 \text{ for all } i \in \{1, \ldots, n\}.$$

Proof: $\Rightarrow$) From Proposition 2.2.4 we obtain that there exist $\mu_1, \ldots, \mu_n \in B_K^-$ such that $x = \sum_{i=1}^n \mu_i e_i$. Then $\sum_{i=1}^n (\lambda_i - \mu_i)e_i = \sum_{i=1}^n \lambda_i e_i - \sum_{i=1}^n \mu_i e_i = x - x = 0$ and hence, by the previous proposition, $|\lambda_i - \mu_i| < 1$ for every $i \in \{1, \ldots, n\}$. Then also $|\lambda_1|, \ldots, |\lambda_n| < 1$.
$\Leftarrow$) $\lambda_i e_i \in A^-$ for every $i \in \{1, \ldots, n\}$ and hence $x = \sum_{i=1}^n \lambda_i e_i \in A^-$. □

2.2.7 Theorem *Let A be a finitely generated B_K-module. Let $\pi : A \to A/A^-$ be the canonical map. Let $e_1, \dots, e_n \in A$. Then:*
$e_1, \dots, e_n$ is a minimally generating collection for A $\iff$ $\pi(e_1), \dots, \pi(e_n)$ is a base for the $\mathbf{k}$-vector space A/A^-.

Proof: $\Rightarrow$) That $[\pi(e_1), \dots, \pi(e_n)] = A/A^-$ follows from the surjectivity of π. We prove that $\pi(e_1), \dots, \pi(e_n)$ are independent. To this end, let $\rho_1, \dots, \rho_n \in \mathbf{k}$ such that $\rho_1\pi(e_1) + \cdots + \rho_n\pi(e_n) = 0$. Let $\lambda_1, \dots, \lambda_n \in B_K$ such that $\rho_\iota = \lambda_\iota + B_K^- \quad (i \in \{1, \dots, n\})$. Then $\lambda_1 e_1 + \cdots + \lambda_n e_n \in A^-$. From the previous proposition we obtain that $\lambda_\iota \in B_K^-$ for every $i \in \{1, \dots, n\}$ and hence, $\rho_1 = \dots = \rho_n = 0$. We see that $\pi(e_1), \dots, \pi(e_n)$ is a base.
$\Leftarrow$) Let $m \in \mathbb{N}$ and $g_1, \dots, g_m \in A$ such that $A = \mathrm{co}\{g_1, \dots, g_m\}$. Let l be the mininimal number among all $k \in \mathbb{N}$ for which there exist $\iota(1), \dots, i(k) \in \{1, \dots, m\}$ such that $A = \mathrm{co}\{g_{\iota(1)}, \dots, g_{\iota(k)}, e_1, \dots, e_n\}$. Suppose $l > 0$. Let $i(1), \dots, \iota(l) \in \{1, \dots, m\}$ such that $A = \mathrm{co}\{g_{\iota(1)}, \dots, g_{\iota(l)}, e_1, \dots, e_n\}$. Then there exist $\rho_1, \dots, \rho_n \in \mathbf{k}$ such that $\pi(g_{\iota(l)}) = \rho_1\pi(e_1) + \cdots + \rho_n\pi(e_n)$. Let $\lambda_1, \dots, \lambda_n \in B_K$ such that $\rho_\iota = \lambda_\iota + B_K^-$ for $i = 1, \dots, n$. Then

$$g_{\iota(l)} + A^- = \sum_{\iota=1}^{n} \lambda_\iota e_\iota + A^-.$$

Hence there exists a $z \in A^-$ such that $g_{\iota(l)} = \sum_{\iota=1}^{n} \lambda_\iota e_\iota + z$. As A is generated by $g_{\iota(1)}, \dots, g_{\iota(l)}, e_1, \dots, e_n$ there exist $\nu_1, \dots, \nu_l, \mu_1, \dots, \mu_n \in B_K^-$ such that $z = \sum_{j=1}^{l} \nu_j g_{\iota(j)} + \sum_{\iota=1}^{n} \mu_\iota e_\iota$. Then

$$(1 - \nu_l) g_{\iota(l)} = \sum_{j=1}^{l-1} \nu_j g_{\iota(j)} + \sum_{\iota=1}^{n} (\lambda_\iota + \mu_\iota) e_\iota.$$

Now $|1 - \nu_l| = 1$ and hence $(1 - \nu_l)^{-1} \in B_K$. Thus,

$$g_{\iota(l)} = \sum_{j=1}^{l-1} (1 - \nu_l)^{-1} \nu_j g_{\iota(j)} + \sum_{\iota=1}^{n} (1 - \nu_l)^{-1} (\lambda_\iota + \mu_\iota) e_\iota.$$

Hence, $g_{\iota(l)} \in \mathrm{co}\{g_{\iota(1)}, \dots, g_{\iota(l-1)}, e_1, \dots, e_n\}$, implying that

$$A = \mathrm{co}\{g_{\iota(1)}, \dots, g_{\iota(l)}, e_1, \dots, e_n\} = \mathrm{co}\{g_{\iota(1)}, \dots, g_{\iota(l-1)}, e_1, \dots, e_n\}.$$

This is in contradiction with the minimality of l.
Hence, $l = 0$ and $A = \mathrm{co}\{e_1, \dots, e_n\}$.
Suppose also $f_1, \dots, f_m \in A$ such that $A = \mathrm{co}\{f_1, \dots, f_m\}$. As π is surjective we obtain that $[\pi(f_1), \dots, \pi(f_m)] = A/A^-$ and from $\dim A/A^- = n$ it follows that $m \geq n$.
We obtain that $e_1, \dots, e_n$ is a minimally generating collection for A. □

2.2.8 Corollary *Let A be a finitely generated B_K-module and $g_1, \dots, g_m \in A$ such that $A = \mathrm{co}\{g_1, \dots, g_m\}$. Then there exist $n \leq m$ and $\iota(1), \dots, i(n) \in \{1, \dots, m\}$ such that $g_{\iota(1)}, \dots, g_{\iota(n)}$ is a minimally generating collection for A.*

Proof: We have that $[\pi(g_1),\dots,\pi(g_m)] = A/A^-$, where $\pi : A \to A/A^-$ is the canonical map. Hence there exist $n \le m$ and $i(1),\dots,i(n) \in \{1,\dots,m\}$ such that $\pi(g_{i(1)}),\dots,\pi(g_{i(n)})$ is a base for the $\mathbf{k}$-vector space A/A^-. By using the previous theorem we obtain that $g_{i(1)},\dots,g_{i(n)}$ is a minimally generating collection for A. □

2.2.9 Corollary *Let A be a finitely generated B_K-module. Let $g_1,\dots,g_m \in A$ such that for every $\lambda_1,\dots,\lambda_m \in B_K$*

$$\lambda_1 g_1 + \cdots + \lambda_m g_m \in A^- \iff \lambda_1,\dots,\lambda_m \in B_K^-.$$

Then there exist $n \ge m$ and $g_{m+1},\dots,g_n \in A$ such that $g_1,\dots,g_n$ is a minimally generating collection for A.

Proof: From the fact that A is finitely generated we obtain that the $\mathbf{k}$-vector space A/A^- is finite-dimensional. Just like in the proof of Theorem 2.2.7 we obtain that $\pi(g_1),\dots,\pi(g_m)$ are independent. Hence, $\pi(g_1),\dots,\pi(g_m)$ can be completed to a base. That is to say that there exist $v_1,\dots,v_k \in A/A^-$ such that $\pi(g_1),\dots,\pi(g_m),v_1,\dots,v_k$ is a base for the $\mathbf{k}$-vector space A/A^-. Let $n = m + k$ and for every $i \in \{1,\dots,k\}$ let $g_{m+i} \in A$ be such that $\pi(g_{m+i}) = v_i$. From Theorem 2.2.7 we obtain that $g_1,\dots,g_n$ is a minimally generating collection for A. □

2.2.10 Proposition *Let A be a finitely generated B_K-module.*
Let $e_1,\dots,e_n \in A$ such that $A = \mathrm{co}\{e_1,\dots,e_n\}$. Then the following two assertions are equivalent.

(i) *$e_1,\dots,e_n$ is a minimally generating collection for A*

(ii) *If $\lambda_1,\dots,\lambda_n \in B_K$ such that $\lambda_1 e_1 + \cdots + \lambda_n e_n = 0$ then $|\lambda_i| < 1$ for every $i \in \{1,\dots,n\}$.*

Proof: (i) ⇒ (ii): This is Proposition 2.2.6.
(ii) ⇒ (i): For every $i \in \{1,\dots,n\}$ we have that

$$e_i \notin \mathrm{co}\{e_1,\dots,e_{i-1},e_{i+1},\dots,e_n\}.$$

By using Corollary 2.2.8 we obtain that $e_1,\dots,e_n$ is a minimally generating collection. □

2.2.11 Remark The following is not true.
Let A be a finitely generated B_K-module and let $g_1,\dots,g_m \in A \setminus A^-$ such that for every $\lambda_1,\dots,\lambda_m \in B_K$

$$\lambda_1 g_1 + \cdots + \lambda_m g_m = 0 \Rightarrow \lambda_1,\dots,\lambda_m \in B_K^-.$$

Then $g_1,\dots,g_m$ can be completed to a minimally generating collection for A. (Compare with Corollary 2.2.9.)

For example, let p be a prime number and $K = \mathbb{Q}_p$. Let e_1,e_2 be the canonical base for K^2 and let $A = \mathrm{co}\{pe_1, e_1 + e_2\}(\subset K^2)$. Let $g_1 = pe_1$ and $g_2 = pe_2$. Then $g_1,g_2 \in A \setminus A^-$ and for every $\lambda_1,\lambda_2 \in B_K$ we have that

$$\lambda_1 g_1 + \lambda_2 g_2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 0.$$

But g_1, g_2 can not be completed to a minimally generating collection for A.

2.2.12 Definition Let A be a B_K-module. Let $n \in \mathbb{N}$ be such that there exist a minimally generating collection consisting of n members of A. Then A is called *n-generated.*

2.2.13 Proposition *Let A be an n-generated B_K-module. Then there exists a minimally generating collection $e_1, \ldots, e_n$ for A and a $k \le n$ such that* $\mathrm{co}\{e_1, \ldots, e_k\}$ *is torsion free,* $A_t = \mathrm{co}\{e_{k+1}, \ldots, e_n\}$ *and*

$$A = \mathrm{co}\{e_1, \ldots, e_k\} \oplus A_t.$$

Proof: Let $\pi : A \to A/A_t$ be the canonical map. Now A/A_t is k-generated for some $k \le n$. Let $u_1, \ldots, u_k$ be a minimally generating collection for A/A_t. Let $e_1, \ldots, e_k \in A$ be such that $u_i = \pi(e_i)$ $(i = 1, \ldots, k)$. Then $A = \mathrm{co}\{e_1, \ldots, e_k\} + A_t$. We prove that this last sum is direct. To this end, let $x \in \mathrm{co}\{e_1, \ldots, e_k\} \cap A_t$. Then there exist $\lambda_1, \ldots, \lambda_k \in B_K$ such that $x = \sum_{i=1}^k \lambda_i e_i$. Now $x \in A_t$ and hence $0 = \pi(x) = \sum_{i=1}^k \lambda_i u_i$. Suppose not all λ_i vanish. By renumeration we may suppose that $|\lambda_1| = \max\{|\lambda_1|, \ldots, |\lambda_k|\}$. Then $\lambda_1^{-1}\lambda_i \in B_K$ for every $i \in \{1, \ldots k\}$ and as A/A_t is torsion free we obtain that $\sum_{i=1}^k (\lambda_1^{-1}\lambda_i)u_i = 0$. Then $u_1 = \sum_{i=2}^k (-\lambda_1^{-1}\lambda_i)u_i \in \mathrm{co}\{u_2, \ldots, u_k\}$. This is in conflict to the minimality of $u_1, \ldots, u_k$. Thus, $\lambda_1 = \cdots = \lambda_k = 0$ and hence $x = 0$. As a special case we have seen that if $\lambda_1, \ldots, \lambda_k \in B_K$ are such that $\lambda_1 e_1 + \cdots + \lambda_k e_k = 0$ then $\lambda_1 = \ldots = \lambda_k = 0$. In particular, $\mathrm{co}\{e_1, \ldots, e_k\}$ is torsion free. We have $A = \mathrm{co}\{e_1, \ldots, e_k\} \oplus A_t$ and hence $A_t \sim A/\mathrm{co}\{e_1, \ldots, e_k\}$. Thus also A_t is finitely generated. Let $e_{k+1}, \ldots, e_n$ be a minimally generating collection for A_t. Suppose that $\lambda_1, \ldots, \lambda_n \in B_K$ are such that $\lambda_1 e_1 + \cdots + \lambda_n e_n = 0$. Then $\lambda_1 e_1 + \cdots + \lambda_k e_k = 0$ and $\lambda_{k+1} e_{k+1} + \cdots + \lambda_n e_n = 0$. Then $\lambda_1 = \ldots = \lambda_k = 0$ and from the fact that $e_{k+1}, \ldots, e_n$ is a minimally generating collection for A_t it follows that $|\lambda_i| < 1$ for every $i \in \{k+1, \ldots, n\}$. From Proposition 2.2.10 we obtain that $e_1, \ldots, e_n$ is a minimally generating collection for A. □

2.2.14 Proposition *Let A be an n-generated B_K-module and let B be a submodule of A. Then one of the following assertions is true.*

(i) *B is not finitely generated.*

(ii) *B is m-generated for some $m \le n$.*

Proof: By induction with respect to n.
1: $n = 1$. Let $y \in A$ such that $A = \mathrm{co}\{y\}$. Suppose that (i) is not true. Then B is finitely generated. Hence there exist $x_1, \ldots, x_m \in B$ such that $B = \mathrm{co}\{x_1, \ldots, x_m\}$. For every $i \in \{1, \ldots, m\}$ there exists a $\lambda_i \in B_K$ such that $x_i = \lambda_i y$. Let $j_0 \in \{1, \ldots, m\}$ such that $|\lambda_{j_0}| = \max\{|\lambda_1|, \ldots, |\lambda_m|\}$. Then $B = \mathrm{co}\{\lambda_{j_0} y\}$.
2. Let $n \ge 2$ and suppose the assertion is true for all $(n-1)$-generated B_K-modules. Let A be an n-generated B_K-module. Let $e_1, \ldots, e_n$ be a minimally generating collection of A. Let B be a submodule of A and suppose (i) is not true. Then B is finitely generated. Let $x_1, \ldots, x_k \in B$ such

that $B = \mathrm{co}\{x_1,\dots,x_k\}$. By using $A = \mathrm{co}\{e_1,\dots,e_n\}$ and Proposition 2.2.6 we can find $y_1,\dots,y_k \in A \setminus A^-$ and $\lambda_1,\dots,\lambda_k \in B_K$ such that $x_i = \lambda_i y_i \quad (i = 1,\dots,k)$. Let $C_i = \{\lambda \in B_K \mid |\lambda| \le |\lambda_i|\} \quad (i = 1,\dots,k)$. Then $B = C_1y_1 + \cdots + C_ky_k$. The C_i's are ordered by inclusion. By renumeration we may suppose that $C_i \subset C_k$ for every $i \in \{1,\dots,k\}$. There exist $\mu_1^1,\dots,\mu_n^1,\dots,\mu_1^k,\dots,\mu_n^k \in B_K$ such that $y_i = \sum_{j=1}^n \mu_j^i e_j \quad (i = 1,\dots,k)$. As $y_k \in A \setminus A^-$ we obtain that there exists a $j_0 \in \{1,\dots,n\}$ such that $|\mu_{j_0}^k| = 1$. By renumeration of $e_1,\dots,e_n$ we may suppose that $j_0 = n$.
Let $z_1,\dots,z_{k-1}$ be defined by $z_i = y_i - (\mu_n^k)^{-1}\mu_n^i y_k \quad (i = 1,\dots,k-1)$. Then $z_i = \sum_{j=1}^n \mu_j^i e_j - \sum_{j=1}^n (\mu_n^k)^{-1}\mu_n^i\mu_j^k e_j \in \mathrm{co}\{e_1,\dots,e_{n-1}\}$ for every $i \in \{1,\dots,k-1\}$.
Furthermore, $B = C_1z_1 + \cdots + C_{k-1}z_{k-1} + C_ky_k$. For let $x \in B$. There exist $\nu_1 \in C_1,\dots,\nu_k \in C_k$ such that $x = \nu_1y_1 + \cdots + \nu_ky_k$. Then

$$x = \nu_1(z_1 + (\mu_n^k)^{-1}\mu_n^1 y_k) + \cdots + \nu_{k-1}(z_{k-1} + (\mu_n^k)^{-1}\mu_n^{k-1}y_k) + \nu_ky_k$$

and thus,

$$x = \nu_1z_1 + \cdots + \nu_{k-1}z_{k-1} + (\nu_k + ((\mu_n^k)^{-1}\mu_n^1\nu_1 + \cdots + (\mu_n^k)^{-1}\mu_n^{k-1}\nu_{k-1}))y_k.$$

Now $C_i \subset C_k$ for every $i \in \{1,\dots,k-1\}$ and hence $(\mu_n^k)^{-1}\mu_n^i\nu_i \in C_k$ for every $i \in \{1,\dots,k-1\}$. Hence, $\nu_k + ((\mu_n^k)^{-1}\mu_n^1\nu_1 + \cdots + (\mu_n^k)^{-1}\mu_n^{k-1}\nu_{k-1}) \in C_k$, which implies that $x \in C_1z_1 + \cdots + C_{k-1}z_{k-1} + C_ky_k$.
We see that $B \subset C_1z_1 + \cdots + C_{k-1}z_{k-1} + C_ky_k$.
In the same way we obtain that $C_1z_1 + \cdots + C_{k-1}z_{k-1} + C_ky_k \subset B$.
Let $\hat{A} = \mathrm{co}\{e_1,\dots,e_{n-1}\}$ and let $\hat{B} = C_1z_1 + \cdots + C_{k-1}z_{k-1}$. Then $\hat{A}$ is $(n-1)$-generated, $\hat{B} \subset \hat{A}$ and (i) is not true. Hence there exist $m \le n-1$ and $w_1,\dots,w_m \in \hat{B}$ such that $\hat{B} = \mathrm{co}\{w_1,\dots,w_m\}$. Then

$$B = \hat{B} + C_ky_k = \mathrm{co}\{w_1,\dots,w_m\} + \mathrm{co}\{\lambda_ky_k\} = \mathrm{co}\{w_1,\dots,w_m,x_k\}.$$

We see that B is $(m+1)$-generated and $m+1 \le n$. □

2.2.15 Proposition *Let E be an n-dimensional K-vector space. Let A be an absolutely convex subset of E. Then one of the following assertions is true.*

(i) *A is as a B_K-module not finitely generated.*

(ii) *A is m-generated for some $m \le n$.*

Proof: Let $e_1,\dots,e_n$ be a vector space base for E. Suppose (i) is not true. Then A is finitely generated. Let $y_1,\dots,y_k \in A$ such that $A = \mathrm{co}\{y_1,\dots,y_k\}$. There exist $\lambda_1^1,\dots,\lambda_n^1,\dots,\lambda_1^k,\dots,\lambda_n^k \in K$ such that $y_i = \sum_{j=1}^n \lambda_j^i e_j \quad (i = 1,\dots,k)$.
Let $\mu \in K$ such that $|\mu| \ge \max\{|\lambda_1^1|,\dots,|\lambda_n^1|,\dots,|\lambda_1^k|,\dots,|\lambda_n^k|\}$ and let $M = \mathrm{co}\{\mu e_1,\dots,\mu e_n\}$. Then M is an n-generated B_K-module and A is a finitely generated submodule of M. From the previous proposition we obtain that there exist an $m \le n$ such that A is m-generated. □

We recall the following definition from [23], 2.1 and 2.5(v).

2.2.16 Definition An absolutely convex subset A of an n-dimensional K-vector space E is called *elementary* if there exist $m \le n$, absolutely convex subsets $C_1, \ldots, C_m$ of K and $x_1, \ldots, x_m \in E$ with $A = C_1x_1 \oplus \cdots \oplus C_mx_m$.

2.2.17 Lemma *Let $n \in \mathbb{N}$ and let E be an n-dimensional K-vector space. Let A be a finitely generated absolutely convex subset of E such that $[A] = E$. Suppose $B \subset A$ is an elementary absolutely convex subset such that also $[B] = E$. Then there exist $x_1, \ldots, x_n \in A \setminus A^-$ and absolutely convex subsets $C_1, \ldots, C_n$ of B_K such that $A = \operatorname{co}\{x_1, \ldots, x_n\}$ and $B = C_1x_1 + \cdots + C_nx_n$.*

Proof: By induction.
For $n = 1$ it is clear.
Let $n > 1$. Let E be a K-vector space with $\dim E = n$. Let A and B be absolutely convex subsets of E such that A is finitely generated, B is elementary, $B \subset A$ and $[B] = [A] = E$. Let $y_1, \ldots, y_n \in A$ such that $A = \operatorname{co}\{y_1, \ldots, y_n\}$. There exist $z_1, \ldots, z_n \in E$ and absolutely convex subsets $B_1, \ldots, B_n$ of K such that $B = B_1z_1 + \ldots + B_nz_n$. As $B \subset A$ and A is finitely generated we may assume that $z_1, \ldots, z_n \in A \setminus A^-$ and $B_1, \ldots, B_n \subset B_K$. There exist $\lambda_1^1, \ldots, \lambda_n^1, \ldots, \lambda_1^n, \ldots, \lambda_n^n \in B_K$ such that

$$z_i = \sum_{j=1}^{n} \lambda_j^i y_j \quad (i = 1, \ldots, n).$$

Now $B_1, \ldots, B_n$ are ordered by inclusion. Hence, there exists an $i_0 \in \{1, \ldots, n\}$ such that $B_j \subset B_{i_0}$ for every $j \in \{1, \ldots, n\}$. As $z_{i_0} \in A \setminus A^-$ there exists a $j_0 \in \{1, \ldots, n\}$ such that $|\lambda_{j_0}^{i_0}| = 1$. By renumeration of $y_1, \ldots, y_n$ and $z_1, \ldots, z_n$ we may suppose that $i_0 = j_0 = n$. Now define $\hat{z}_1, \ldots, \hat{z}_{n-1}$ by $\hat{z}_i = z_i - (\lambda_n^n)^{-1}\lambda_n^i z_n \quad (i = 1, \ldots, n)$.
In the same way as in the proof of Proposition 2.2.14 we obtain that

$$\hat{z}_1, \ldots, \hat{z}_{n-1} \in \operatorname{co}\{y_1, \ldots, y_{n-1}\}$$

and $B = B_1\hat{z}_1 + \ldots + B_{n-1}\hat{z}_{n-1} + B_nz_n$. As $[B_1\hat{z}_1 + \ldots + B_{n-1}\hat{z}_{n-1} + B_nz_n] = [B] = [y_1, \ldots, y_n]$ and $[B_1\hat{z}_1 + \ldots + B_{n-1}\hat{z}_{n-1}] \subset [y_1, \ldots, y_{n-1}]$ we obtain that $[B_1\hat{z}_1 + \ldots + B_{n-1}\hat{z}_{n-1}] = [y_1, \ldots, y_{n-1}]$. Let $\hat{E} = [y_1, \ldots, y_{n-1}]$, $\hat{A} = \operatorname{co}\{y_1, \ldots, y_{n-1}\}$ and $\hat{B} = B_1\hat{z}_1 + \ldots + B_{n-1}\hat{z}_{n-1}$. Then $\dim \hat{E} = n - 1$, $\hat{B}$ is an elementary subset of $\hat{E}$, $\hat{A}$ is a finitely generated subset of $\hat{E}$, $\hat{B} \subset \hat{A}$ and $[\hat{B}] = [\hat{A}] = \hat{E}$. By induction there exist $x_1, \ldots, x_{n-1} \in A \setminus A^-$ and absolutely convex subsets $C_1, \ldots, C_{n-1}$ of B_K such that $\hat{A} = \operatorname{co}\{x_1, \ldots, x_{n-1}\}$ and $\hat{B} = C_1x_1 + \cdots + C_{n-1}x_{n-1}$.
Let $C_n = B_n$ and $x_n = z_n$. Then $x_1, \ldots, x_n \in A \setminus A^-$, $C_1, \ldots, C_n$ are absolutely convex subsets of B_K and $B = C_1x_1 + \cdots + C_nx_n$.
Now $A = \operatorname{co}\{x_1, \ldots, x_n\}$ as we see as follows. It is obvious that $\operatorname{co}\{x_1, \ldots, x_n\} \subset A$. Suppose $v \in A$. Then there exist $\mu_1, \ldots, \mu_n \in B_K$ such that $v = \mu_1y_1 + \ldots + \mu_ny_n$. Then v equals

$$\sum_{j=1}^{n-1} \mu_jy_j + \mu_n(\lambda_n^n)^{-1}(z_n - \sum_{j=1}^{n-1} \lambda_j^n y_j) = \sum_{j=1}^{n-1} \mu_j - \mu_n(\lambda_n^n)^{-1}\lambda_j^n y_j + \mu_n(\lambda_n^n)^{-1}z_n$$

and hence $v \in \operatorname{co}\{y_1, \ldots, y_{n-1}\} + \operatorname{co}\{z_n\} = \operatorname{co}\{x_1, \ldots, x_n\}$. □

2.2.18 Remark From the previous lemma one can prove the following.
Let E be a finite dimensional K-vector space and let $\| \ \|_1$ and $\| \ \|_2$ be vector space norms (see Definition 3.3.3) on E, such that there exist orthogonal bases with respect to these norms. Then there exists a base of E that is orthogonal with respect to both $\| \ \|_1$ and $\| \ \|_2$.
(Compare [19], Theorem 1.11 applied to $E = (E, \| \ \|_1)$, $F = (E, \| \ \|_2)$ and $T = \mathrm{id}_E$.)

2.2.19 Theorem *Let K be spherically complete. Let A be an n-generated B_K-module. Then there exist $x_1, \dots, x_n \in A$ such that $A = \mathrm{co}\{x_1, \dots, x_n\}$ and*

$$\textit{if } \lambda_1, \dots, \lambda_n \in B_K \textit{ such that } \sum_{i=1}^{n} \lambda_i x_i = 0 \textit{ then } \lambda_1 x_1 = \dots = \lambda_n x_n = 0.$$

Proof: According to Proposition 2.2.13 there exist a minimally generating collection $e_1, \dots, e_n$ for A and $k \le n$ such that $\mathrm{co}\{e_1, \dots, e_k\}$ is torsion free, $A_t = \mathrm{co}\{e_{k+1}, \dots, e_n\}$ and $A = \mathrm{co}\{e_1, \dots, e_k\} \oplus A_t$.
Then, in particular, $\mathrm{co}\{e_1, \dots, e_k\}$ is k-generated. Thus if $\lambda_1, \dots, \lambda_k \in B_K$ such that $\lambda_1 e_1 + \cdots + \lambda_k e_k = 0$ then $\lambda_1 = \dots = \lambda_k = 0$ (see the proof of Proposition 2.2.13.
Observe that if $n = k$, we can take $x_1 = e_1, \dots, x_n = e_n$, so assume $n > k$. Furthermore, $E = K^{n-k}$ is an $n-k$-dimensional vector space. Let $f_1, \dots, f_{n-k}$ be the canonical base for E. Let $C = \mathrm{co}\{f_1, \dots, f_{n-k}\}$. We define a homomorphism $\varphi : C \to A_t$ by

$$\varphi\Big(\sum_{i=1}^{n-k} \lambda_i f_i\Big) = \sum_{i=k+1}^{n} \lambda_{i-k} e_i \quad (\lambda_1, \dots, \lambda_{n-k} \in B_K).$$

Then φ surjective and $[\mathrm{Ker}\,\varphi] = [C] = E$. By [23], Corollary 2.13, $\mathrm{Ker}\,\varphi$ is elementary. Hence there exist $y_1, \dots, y_{n-k} \in C \setminus C^-$ and absolutely convex subsets $D_1, \dots, D_{n-k}$ of B_K such that $C = \mathrm{co}\{y_1, \dots, y_{n-k}\}$ and $\mathrm{Ker}\,\varphi = D_1 y_1 + \cdots + D_{n-k} y_{n-k}$. Then A_t is generated by $\{\varphi(y_1), \dots, \varphi(y_{n-k})\}$ and if $\lambda_1, \dots, \lambda_{n-k} \in B_K$ such that $\lambda_1 \varphi(y_1) + \cdots + \lambda_{n-k} \varphi(y_{n-k}) = 0$ then $\varphi(\lambda_1 y_1 + \cdots + \lambda_{n-k} y_{n-k}) = 0$ and hence $\lambda_1 y_1 + \cdots + \lambda_{n-k} y_{n-k} \in \mathrm{Ker}\,\varphi = D_1 y_1 + \cdots + D_{n-k} y_{n-k}$. As $y_1, \dots, y_{n-k}$ are independent this implies that $\lambda_1 \in D_1, \dots, \lambda_{n-k} \in D_{n-k}$. Hence, $\lambda_1 y_1, \dots, \lambda_{n-k} y_{n-k} \in \mathrm{Ker}\,\varphi$ and thus $\lambda_1 \varphi(y_1) = \dots = \lambda_{n-k} \varphi(y_{n-k}) = 0$.
Then $A = \mathrm{co}\{e_1, \dots, e_k, \varphi(y_1), \dots, \varphi(y_{n-k})\}$ and if $\lambda_1, \dots, \lambda_n \in B_K$ are such that $\lambda_1 e_1 + \cdots + \lambda_k e_k + \lambda_{k+1} \varphi(y_1) + \cdots + \lambda_n \varphi(y_{n-k}) = 0$ then $\lambda_1 e_1 + \cdots + \lambda_k e_k = 0$ and $\lambda_{k+1} \varphi(y_1) + \cdots + \lambda_n \varphi(y_{n-k}) = 0$. This implies that $\lambda_1 = \dots = \lambda_k = 0$ and $\lambda_{k+1} \varphi(y_{k+1}) = \dots = \lambda_n \varphi(y_{n-k}) = 0$.
Take $x_i = e_i$ for $i \in \{1, \dots, k\}$ and $x_i = \varphi(y_{i-k})$ for $i \in \{k+1, \dots, n\}$. □

2.2.20 Remark For the proof of this theorem we need the spherically completeness of K only to ensure that $\mathrm{Ker}\,\varphi$ is elementary. Hence, we also have the following.
Let A be an n-generated B_K-module such that A_t is m-generated. Suppose there exists a finitely generated absolutely convex subset C of K^m and a surjective homomorphism $\varphi : C \to A_t$ such that $\mathrm{Ker}\,\varphi$ *is elementary. Then there exists $x_1, \dots, x_n$ as in the above theorem.*

2.2.21 Corollary *Let K be spherically complete and let A be an n-generated B_K-module. Then there exist $k \le n$ and absolutely convex subsets $C_{k+1}, \dots, C_n$ of B_K such that $A \sim B_K^k \times B_K/C_{k+1} \times \dots \times B_K/C_n$.*

Proof: By Theorem 2.2.19 there exist $k \le n$ and $x_1, \dots, x_n \in A$ such that $A = \mathrm{co}\{x_1, \dots, x_n\}$ and

$$\text{if } \lambda_1, \dots, \lambda_n \in B_K \text{ such that } \sum_{i=1}^{n} \lambda_i x_i = 0 \text{ then } \lambda_1 x_1 = \dots = \lambda_n x_n = 0.$$

For every $i \in \{k+1, \dots, n\}$ let $C_i = \{\lambda \in B_K \mid \lambda x_i = 0\}$. Then

$$A = \mathrm{co}\{x_1\} \oplus \cdots \oplus \mathrm{co}\{x_k\} \oplus \mathrm{co}\{x_{k+1}\} \oplus \cdots \oplus \mathrm{co}\{x_n\} \sim$$
$$\bigoplus_{i=1}^{k} B_K \oplus B_K/C_{k+1} \oplus \cdots \oplus B_K/C_n \sim B_K^k \times B_K/C_{k+1} \times \dots \times B_K/C_n.$$

□

2.2.22 Remark The spherically completeness of K in Corollary 2.2.21 can not be omitted. To show this, we will construct a 2-generated B_K-module A such that $A = \mathrm{co}\{u_1\} \oplus \mathrm{co}\{u_2\}$ for no $u_1, u_2 \in A$.
Let K be not spherically complete. Let ν be a norm on K^2 such that K^2 has no orthogonal base with respect to ν. We assume that $\nu(K^2) = |K|$. In the remarks after Lemma 3.14 in [20] it is shown that such a norm exists.
Let e_1, e_2 be the canonical base for K^2. Every two norms on K^2 are equivalent. Let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence in B_K. Now $(\mu_1 e_1 + \mu_2 e_2) \mapsto |\mu_1| \vee |\mu_2|$ is a norm on K^2 and $1 \vee |\lambda_n| \not\to 0$ and $|\lambda_n| \vee 1 \not\to 0$ $(n \to \infty)$. Hence, $\nu(e_1 + \lambda_n e_2) \not\to 0$ and $\nu(\lambda_n e_1 + e_2) \not\to 0$ $(n \to \infty)$. We obtain that there exists an $\varepsilon > 0$ such that $\nu(e_1 + \lambda e_2) > \varepsilon$ and $\nu(\lambda e_1 + e_2) > \varepsilon$ for every $\lambda \in B_K$. Let $\mu \in K$ with $|\mu| > \frac{1}{\varepsilon}$.
Let $\mathrm{B}_\nu(0,1) = \{x \in K^2 \mid \nu(x) \le 1\}$. Let $A = \mathrm{co}\{\mu e_1, \mu e_2\}/\mathrm{B}_\nu(0,1)$. As not $\mu e_1 \in \mu e_2 + \mathrm{B}_\nu(0,1)$ and not $\mu e_2 \in \mu e_1 + \mathrm{B}_\nu(0,1)$ we obtain that $A = \mathrm{co}\{\mu e_1 + \mathrm{B}_\nu(0,1), \mu e_2 + \mathrm{B}_\nu(0,1)\}$ is 2-generated.
Let $u_1, u_2 \in A$ such that $A = \mathrm{co}\{u_1, u_2\}$. Let $z_1, z_2 \in \mathrm{co}\{\mu e_1, \mu e_2\}$ such that $u_1 = z_1 + \mathrm{B}_\nu(0,1)$ and $u_2 = z_2 + \mathrm{B}_\nu(0,1)$. Then $\nu(z_1), \nu(z_2) > 1$. By symmetry we may assume that $\nu(z_1) \ge \nu(z_2)$. As z_1, z_2 is not an orthogonal base for K^2 with respect to ν there exists a $\lambda \in B_K$ such that $\nu(\lambda z_1 + z_2) < \max(\nu(\lambda z_1), \nu(z_2))$. Then $\nu(\lambda z_1) = \nu(z_2)$. Let $\rho \in B_K$ such that $\nu(z_2)^{-1} < |\rho| \le \nu(\lambda z_1 + z_2)^{-1}$. Then $\nu(\rho\lambda z_1 + \rho z_2) \le 1$ and hence $\rho\lambda u_1 + \rho u_2 = 0$. But $\nu(\rho z_2) > 1$ and hence $\rho u_2 \ne 0$.
We obtain that not $A = \mathrm{co}\{u_1\} \oplus \mathrm{co}\{u_2\}$.
We see that, according to Remark 2.2.20, $B_\nu(0,1)$ apparently is not elementary.

B_K-modules of Finite Rank

2.2.23 Definition A B_K-module A is called (a *B_K-module*) *of finite rank* if there exist $n \in \mathbb{N}$, an absolutely convex subset B of K^n and a surjective homomorphism $B \to A$.

The class of all B_K-modules of finite rank is denoted by $\mathcal{F}_K$.

2.2.24 Remark If $|K|$ is trivial then $\mathcal{F}_K$ coincides with the class of all finite-dimensional K-vector spaces.

2.2.25 Examples Let the valuation on K be non-trivial.
A K-vector space E is a member of $\mathcal{F}_K$ if and only if it is finite dimensional. Furthermore, every absolutely convex subset of a finite dimensional K-vector space is a member of $\mathcal{F}_K$.
A non-trivial member of $\mathcal{F}_K$ is, for instance, K/B_K.

2.2.26 Proposition *$\mathcal{F}_K$ is the smallest class C of B_K-modules such that*

1) *$K \in C$.*

2) *C is closed with respect to finite direct sums.*

3) *C is closed with respect to submodules.*

4) *C is closed with respect to homomorphic images.*

Proof: We first show that $C = \mathcal{F}_K$ has properties 1), 2), 3) and 4).
1) As $K \subset K$ and the identity map on K is a surjective homomorphism we obtain that $K \in \mathcal{F}_K$.
2) Let $N \in \mathbb{N}$ and $A_1, \ldots, A_N \in \mathcal{F}_K$. There exist $k_1, \ldots, k_N \in \mathbb{N}$, absolutely convex subsets $B_1 \subset K^{k_1}, \ldots, B_N \subset K^{k_N}$ and surjective homomorphisms $\varphi_1 : B_1 \to A_1, \ldots, \varphi_N : B_N \to A_N$. Let $m = \sum_{\iota=1}^N k_\iota$. Then $\bigoplus_{\iota=1}^N K^{k_\iota} \sim K^m$ and hence $B := \bigoplus_{\iota=1}^N B_\iota \sim C$ for some absolutely convex subset C of K^m. Let $\rho : C \to B$ be a bijective homomorphism and let $\varphi : B \to \bigoplus_{\iota=1}^N A_\iota$ be defined by

$$\varphi((x_1, \ldots, x_N)) = (\varphi_1(x_1), \ldots, \varphi_N(x_N)) \quad (x_1 \in B_1, \ldots, x_N \in B_N).$$

Then φ is a surjective homomorphism. Then $\varphi \circ \rho$ is a surjective homomorphism $C \to \bigoplus_{\iota=1}^N A_\iota$. Hence, $\bigoplus_{\iota=1}^N A_\iota \in \mathcal{F}_K$.
3) Let $A \in \mathcal{F}_K$ and let B be a submodule of A. There exist $n \in \mathbb{N}$, an absolutely convex subset C of K^n and a surjective homomorphism $\varphi : C \to A$. Then $\varphi^{-1}(B)$ is an absolutely convex subset of K^n and $\varphi|\varphi^{-1}(B)$ is a surjective homomorphism $\varphi^{-1}(B) \to B$. Hence, $B \in \mathcal{F}_K$.
4) Let $A \in \mathcal{F}_K$ and let B be a B_K-module such that there exists a surjective homomorphism $\rho : A \to B$. There exist $n \in \mathbb{N}$, an absolutely convex subset C of K^n and a surjective homomorphism $\varphi : C \to A$. Then $\rho \circ \varphi : C \to B$ is a surjective homomorphism and hence $B \in \mathcal{F}_K$.
Now Suppose C is a class of B_K-modules that has the properties 1), 2), 3) and 4). We show that $\mathcal{F}_K \subset C$. In fact:
(i) $K \in C$ and $K^n = \bigoplus_{\iota=1}^n K$. With 2) we obtain that $K^n \in C$.
(ii) If $n \in \mathbb{N}$ and B is an absolutely convex subset of K^n then, by (i), $K^n \in C$ and from 3) we obtain that $B \in C$.
(iii) Let $A \in \mathcal{F}_K$. Then there exist $n \in \mathbb{N}$, an absolutely convex subset B of K^n and a surjective homomorphism $\varphi : B \to A$. From (ii) we obtain that $B \in C$ and from 4) it follows that $A \in C$.
We see $\mathcal{F}_K \subset C$. □

2.2.27 Definition Let A be a B_K-module of finite rank. Then the *rank* of A is the minimal number among all $n \in \mathbb{N}$ for which there exists an absolutely convex subset B of K^n and a surjective homomorphism $B \to A$. The rank of A is denoted by $\operatorname{rank} A$.

2.2.28 Proposition *Let $A \in \mathcal{F}_K$. Let $n = \operatorname{rank} A$. Let B be an absolutely convex subset of K^n and let $\varphi : B \to A$ be a surjective homomorphism. Then $[B] = K^n$ and for every $y \in K^n$ with $y \neq 0$ there exists a $z \in [y] \cap B$ such that $\varphi(z) \neq 0$.*

Proof: Suppose the assertion is not true. Then there exists a $y \in K^n$, $y \neq 0$ such that $\varphi(z) = 0$ for every $z \in [y] \cap B$. Let $\pi : K^n \to K^n/[y]$ be the canonical map. We have that $[y] \cap B \subset \operatorname{Ker} \varphi$ and, by Corollary 2.1.28, φ induces a surjective homomorphism from $B/([y] \cap B)$ to A. From Theorem 2.1.25 we obtain that $B/([y] \cap B) \sim \pi(B)$. Now $K^n/[y] \sim K^{n-1}$ and hence $\pi(B) \sim C$ for some absolutely convex subset C of K^{n-1}. This implies that there exists a surjective homomorphism $C \to A$ which is in contradiction with $\operatorname{rank} A = n$. □

2.2.29 Remark For a torsion free B_K-module A it is not hard to prove that $\operatorname{rank} A = \dim K \otimes_{B_K} A$.
In [10] modules of finite *Fleischer rank* over valuation domains are defined. For a B_K-module A this definition sounds as follows. If A is torsion free then A is called of finite Fleischer rank if $\dim K \otimes_{B_K} A < \infty$. A general B_K-module is called of finite Fleischer rank if there exists a torsion free B_K-module of finite Fleischer rank having A as homomorphic image.
The Fleischer rank of a torsion free B_K-module A of finite Fleischer rank is defined by $\dim K \otimes_{B_K} A$. The Fleischer rank of a general B_K-module A of finite rank is defined as the minimum among all numbers n such that there exists a torsion free B_K-module of Fleischer rank n having A as homomorphic image.
It is not hard to see that, for B_K-modules, the rank that we have introduced here is the same as the Fleischer rank in [10].

2.2.30 Definition If the valuation on K is non-trivial, then an absolutely convex subset B of a finite dimensional K-vector space E is called *bounded* if for every absorbing submodule U of E there exists a $\lambda \in B_K \setminus \{0\}$ such that $\lambda B \subset U$.
If the valuation on K is trivial then every absolutely convex subset of a finite dimensional K-vector space is bounded.

This notion of boundedness coincides with boundedness in the sense of Definition 4.3.1 and Definition 4.3.3, with respect to the unique locally convex Hausdorff topology on E.

If the valuation on K is non-trivial then it is not hard to prove that for an absolutely convex subset B of K^n: *B is bounded $\Longleftrightarrow$ B contains no linear subspaces apart from $\{0\}$.*

2.2.31 Definition Let A be a B_K-module of finite rank. Then A is called *bounded* if A is a homomorphic image of a bounded absolutely convex subset of K^n for some $n \in \mathbb{N}$.

The class of all bounded B_K-modules of finite rank is denoted by $\mathcal{B}_K$.

2.2.32 Proposition *Let $A \in \mathcal{B}_K$. Let U be an absorbing subset of A. Then there exists a $\lambda \in B_K \setminus \{0\}$ such that $\lambda A \subset U$.*

Proof: There exist $n \in \mathbb{N}$, a bounded absolutely convex subset B of K^n and a surjective homomorphism $\varphi : B \to A$. Now $\varphi^{-1}(U)$ is an absorbing submodule of B, and hence also an absorbing submodule of $[B]$. Since B is bounded in $[B]$ there exists a $\lambda \in B_K \setminus \{0\}$ such that $\lambda B \subset \varphi^{-1}(U)$. Then $\lambda A = \lambda\varphi(B) = \varphi(\lambda B) \subset \varphi(\varphi^{-1}(U)) = U$. □

2.2.33 Proposition *Let $A \in \mathcal{B}_K$ and let $n = \operatorname{rank} A$. Let B be an absolutely convex subset of K^n such that there exists a surjective homomorphism $\varphi : B \to A$. Then B is bounded.*

Proof: Let U be an absorbing submodule of K^n. Then $U \cap B$ is absorbing in B and hence $\varphi(U \cap B)$ is absorbing in A. Thus there exists a $\lambda \in B_K \setminus \{0\}$ such that $\lambda A \subset \varphi(U \cap B)$. Then $\lambda B \subset \varphi^{-1}(\varphi(U \cap B)) = U \cap B + \operatorname{Ker}\varphi$. From Proposition 2.2.28 it follows that $\operatorname{Ker}\varphi$ contains no non-trivial linear subspaces and is therefore bounded. Thus, there exists a $\mu \in B_K \setminus \{0\}$ such that $\mu\operatorname{Ker}\varphi \subset U$. Then $\mu\lambda \in B_K \setminus \{0\}$ and

$$(\mu\lambda)B = \mu(\lambda B) \subset \mu(U \cap B + \operatorname{Ker}\varphi) = \mu(U \cap B) + \mu\operatorname{Ker}\varphi \subset U + U = U.$$

We see that B is bounded. □

The proof of the following proposition is in the same spirit as the one of Proposition 2.2.26 and will be omitted.

2.2.34 Proposition *$\mathcal{B}_K$ is the smallest class C of B_K-modules such that*

1) *$B_K \in C$.*

2) *C is closed with respect to finite direct sums.*

3) *C is closed with respect to submodules.*

4) *C is closed with respect to homomorphic images.*

2.2.35 Proposition *Let A be an n-generated B_K-module. Then $A \in \mathcal{B}_K$ and* $\operatorname{rank} A = n$.

Proof: Let $x_1, \dots, x_n$ be a minimally generating collection for A. Let $e_1, \dots, e_n$ be the canonical base for K^n. Let $B = \operatorname{co}\{e_1, \dots, e_n\}$. Let $\varphi : B \to A$ be defined by $\varphi(\lambda_1 e_1 + \cdots + \lambda_n e_n) = \lambda_1 x_1 + \cdots + \lambda_n x_n \quad (\lambda_1, \dots, \lambda_n \in B_K)$. Then B is bounded and φ is a surjective homomorphism. Hence, $A \in \mathcal{B}_K$ and $\operatorname{rank} A \leq n$.
Now let $m \in \mathbb{N}$, let C be an absolutely convex subset of K^m and let $\vartheta : C \to A$

be a surjective homomorphism. We prove that $m \geq n$.
Let $y_1, \dots, y_n \in C$ be such that $\vartheta(y_i) = x_i$ $(i \in \{1, \dots, n\})$. Let $\lambda_1, \dots, \lambda_n \in K$ be such that $\lambda_1 y_1 + \cdots + \lambda_n y_n = 0$. Suppose that not all λ_i vanish. Let $i \in \{1, \dots, n\}$ be such that $|\lambda_i| = \max\{|\lambda_1|, \dots, |\lambda_n|\}$. Then $y_i = \sum_{j \neq i}(\lambda_i^{-1}\lambda_j)y_j$. and hence $\varphi(y_i) = \sum_{j \neq i}(\lambda_i^{-1}\lambda_j)\varphi(y_j)$. Hence, $x_i \in \operatorname{co}\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}$. This is in conflict to the minimality of $x_1, \dots, x_n$. Thus, $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$. We obtain that the vectors $y_1, \dots, y_n$ are linearly independent and hence $m \geq n$. □

2.2.36 Proposition *Let $A \in \mathcal{F}_K$. Let B be a B_K-module and let $j : B \to A$ be an injective homomorphism. Then $B \in \mathcal{F}_K$. If $A \in \mathcal{B}_K$, then $B \in \mathcal{B}_K$.*

Proof: We have that $B \sim C$ for some submodule C of A. Now $C \in \mathcal{F}_K$ according to 3) of Proposition 2.2.26. Then, by 4) of the same proposition, also $B \in \mathcal{F}_K$. If $A \in \mathcal{B}_K$, then $C \in \mathcal{B}_K$ and hence $B \in \mathcal{B}_K$. □

2.2.37 Proposition *Let $n \in \mathbb{N}$. Let A be an absolutely convex subset of K^n. Let $\lambda \in K$ with $|\lambda| > 1$.*

(i) *There exists an elementary set X such that $A \subset X \subset \lambda A$.*

(ii) *If A is bounded there exists an elementary set X that is m-generated as a B_K-module for some $m \leq n$ such that $A \subset X \subset \lambda A$.*

(iii) *If K is spherically complete then A is elementary.*

Proof: (i) This is Lemma 4.6 in [23].
(ii) Suppose that the valuation on K is discrete. From (i) we obtain that there exists an elementary set X such that $A \subset X \subset \lambda A$. There exist $m \leq n$, absolutely convex subset $C_1, \dots, C_m$ of K and $x_1, \dots, x_m \in K^n$ such that $X = C_1 x_1 \oplus \cdots \oplus C_m x_m$. As A is bounded we obtain that each C_i is bounded. Hence there exist $\lambda_1, \dots, \lambda_m \in K$ such that $C_i = \lambda_i B_K$ $(i = 1, \dots, m)$. Then $X = B_K(\lambda_1 x_1) + \ldots + B_K(\lambda_m x_m) = \operatorname{co}\{\lambda_1 x_1, \dots, \lambda_m x_m\}$.
Suppose that the valuation on K is dense. Let $\mu, \nu \in K$ such that $|\mu|, |\nu| > 1$ and $\lambda = \mu\nu$. By (i) there exists an elementary set Y such that $A \subset Y \subset \mu A$. Let $m \leq n$, $C_1, \dots, C_m$ absolutely convex subsets of K and $x_1, \dots, x_m \in K^n$ be such that $Y = C_1 x_1 \oplus \cdots \oplus C_m x_m$. Now each C_i is bounded and hence for every $i \in \{1, \dots, m\}$ there exists a $\lambda_i \in K$ such that $C_i \subset \lambda_i B_K \subset \nu C_i$. Then $A \subset Y \subset \operatorname{co}\{\lambda_1 x_1, \dots, \lambda_m x_m\} \subset \nu Y \subset \nu(\mu A) = (\nu\mu)A = \lambda A$. We take $X = \operatorname{co}\{\lambda_1 x_1, \dots, \lambda_m x_m\}$.
(iii) This is Corollary 2.13 in [23]. □

2.2.38 Proposition *Let the valuation on K be discrete. Let $A \in \mathcal{B}_K$. Then A is finitely generated.*

Proof: There exist $n \in \mathbb{N}$, a bounded absolutely convex subset B of K^n and a surjective homomorphism $\varphi : B \to A$. From (iii) of Proposition 2.2.37 we obtain that B is elementary and in the proof of (ii) of the same proposition we have seen that B is finitely generated. Hence, $A = \varphi(B)$ is also finitely generated. □

2.2.39 Remark If the valuation on K is dense then B_K^- is a member of $\mathcal{B}_K$ that is not finitely generated.

2.2.40 Proposition *Let $A \in \mathcal{B}_K$. Let $n = \operatorname{rank} A$. If $|K|$ is discrete let $\lambda = 1$, if $|K|$ is dense let $\lambda \in B_K^-$. Then there exists an n-generated B_K-module Y and an injective homomorphism $i : A \to Y$ such that $\lambda Y \subset i(A)$.*

Proof: If $|K|$ is discrete this is Proposition 2.2.38, so suppose $|K|$ is dense. Let B be an absolutely convex subset of K^n and let $\varphi : B \to A$ be a surjective homomorphism. By Proposition 2.2.33, B is bounded and hence there exists an elementary set X that is finitely generated as a B_K-module such that $B \subset X \subset \lambda B$. By Theorem 2.1.25 there exists a bijective homomorphism $\rho : A \to B/\operatorname{Ker}\varphi$. Let $j : B/\operatorname{Ker}\varphi \to X/\operatorname{Ker}\varphi$ be the inclusion map. Then $j \circ \rho : A \to X/\operatorname{Ker}\varphi$ is an injective homomorphism. Furthermore, $\lambda(X/\operatorname{Ker}\varphi) \subset j(B/\operatorname{Ker}\varphi) = j(\rho(A)) = j \circ \rho(A)$.
Now X is n-generated and hence $X/\operatorname{Ker}\varphi$ is m-generated for some $m \le n$. As A is embeddable in $X/\operatorname{Ker}\varphi$ we obtain $m = n$.
Take $Y = X/\operatorname{Ker}\varphi$ and $i = j \circ \rho$. □

2.2.41 Proposition *Let $A \in \mathcal{B}_K$ and let $\operatorname{rank} A = n$. If $|K|$ is discrete let $\lambda = 1$, if $|K|$ is dense let $\lambda \in B_K^-$. Then there exist an $m \le n$ and an m-generated submodule X of A such that $\lambda A \subset X \subset A$.*

Proof: For $|K|$ discrete this is Proposition 2.2.38, so suppose $|K|$ is dense. There exists a bounded absolutely convex subset B of K^n and a surjective homomorphism $\varphi : B \to A$. There exists an elementary set Y that is finitely generated as a B_K-module such that $B \subset Y \subset \lambda^{-1}B$. From Proposition 2.2.28 we obtain that Y must be n-generated. Then also λY is n-generated and $\lambda B \subset \lambda Y \subset \lambda(\lambda^{-1}B) = B$. Then $X := \varphi(\lambda Y)$ is m-generated for some $m \le n$ and $\lambda A \subset \lambda\varphi(Y) = X \subset A$. □

2.2.42 Theorem *Let K be spherically complete. Let $A \in \mathcal{F}_K$. Let $n = \operatorname{rank} A$. Then there exist submodules $A_1, \ldots, A_n$ of A with $\operatorname{rank} A_i = 1$ for each $i \in \{1, \ldots, n\}$ such that $A = A_1 + \cdots + A_n$.*

Proof: There exist an absolutely convex subset B of K^n and a surjective homomorphism $\varphi : B \to A$. From (iii) of Proposition 2.2.37 we obtain that B is elementary. Hence there exist $m \le n$, absolutely convex subsets $C_1, \ldots, C_m$ of K and $x_1, \ldots, x_m \in K^n \setminus \{0\}$ such that $B = C_1 x_1 \oplus \cdots \oplus C_m x_m$. From Proposition 2.2.28 we obtain that $[B] = K^n$, hence $m = n$ and all the C_i's are non-trivial. Now $\operatorname{rank} C_i x_i = 1$ for every $i \in \{1, \ldots, n\}$ and again from Proposition 2.2.28 we obtain that $\varphi(C_i x_i) \neq \{0\}$ and hence $\operatorname{rank} \varphi(C_i x_i) = 1$ for every $i \in \{1, \ldots, n\}$. Then $A = \varphi(C_1 x_1 \oplus \cdots \oplus C_n x_n) = \varphi(C_1 x_1) + \cdots + \varphi(C_n x_n)$. □

We do not know whether a rank n module over a spherically complete B_K is also a *direct* sum of rank 1 submodules.

We now prove that a B_K-module of finite rank is countably generated. First we prove this for B_K-modules of rank 1.

2.2.43 Proposition *A B_K-module of rank 1 is countably generated.*

Proof: Let A be a B_K-module of rank 1. There exist an absolutely convex subset C of K and a surjective homomorphism $\varphi : C \to A$. It is not hard to prove that C is countably generated. Then also $A = \varphi(C)$ is countably generated. □

2.2.44 Proposition *Let $A \in \mathcal{F}_K$. Then A is countably generated.*

Proof: There exist $n \in \mathbb{N}$, an absolutely convex subset B of K^n and a surjective homomorphism $\varphi : B \to A$. We prove that B is countably generated.
If the valuation on K is discrete then B is elementary. Hence there exist $m \le n, x_1, \dots, x_m \in B$ and absolutely convex subsets $C_1, \dots, C_m$ of K such that $B = C_1x_1 + \cdots + C_mx_m$. Now $\operatorname{rank} C_ix_i \le 1$ for every $i \in \{1, \dots, m\}$ and hence each C_ix_i is countably generated. Then also B is countably generated.
Suppose $|K|$ is dense. Let $\lambda_1, \lambda_2, \lambda_3, \dots \in B_K^-$ such that $|\lambda_1| < |\lambda_2| < \dots$ and $\lim_{k\to\infty} |\lambda_k| = 1$. For all $k \ge 1$ there exists an elementary set X_k such that $\lambda_k B \subset X_k \subset B$ for all $k \ge 1$. As each X_k is the sum of finitely many modules of rank 1 we obtain that each X_k is countably generated and hence so is $\bigcup_{k\ge1} X_k$. By applying Lemma 2.8 in [23] we obtain an $m \le n$ and $x_1, \dots, x_m \in B \setminus B^-$ such that $B = B^- + \operatorname{co}\{x_1, \dots, x_m\}$. As $B^- \subset \bigcup_{k\ge1} X_k$ we obtain that $B = \bigcup_{k\ge1} X_k + \operatorname{co}\{x_1, \dots, x_m\}$ and hence B is countably generated. As $A = \varphi(B)$ we obtain that A is also countably generated. □

2.2.45 Remark Let $\mathcal{A}$ denote the collection of all finitely generated B_K-modules and let $\mathcal{B}$ be the collection of all countably generated B_K-modules. We have seen that $\mathcal{A} \subset \mathcal{B}_K \subset \mathcal{F}_K \subset \mathcal{B}$.
If the valuation on K is dense then all these inclusions are strict:
$B_K^- \in \mathcal{B}_K \setminus \mathcal{A}$, $K \in \mathcal{F}_K \setminus \mathcal{B}_K$ and $K^{(\mathbb{N})} \in \mathcal{B} \setminus \mathcal{F}_K$.
If the valuation on K is discrete and non-trivial then the first inclusion is an equality, and the next two inclusions are strict:
$K \in \mathcal{F}_K \setminus \mathcal{B}_K$ and $K^{(\mathbb{N})} \in \mathcal{B} \setminus \mathcal{F}_K$.
Finally, if $|K|$ is trivial even the first two inclusions are equalities and the last inclusion is strict: $K^{(\mathbb{N})} \in \mathcal{B} \setminus \mathcal{F}_K$.

2.3 Linear Compactness

Convex Filters

2.3.1 Definition Let C be a collection of non-empty sets. C is said to have the *finite intersection property* if every intersection of finitely many members of C is non-empty.

2.3.2 Definition Let A be a B_K-module. A subset C of A is called *convex* if $\lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3 \in A$ for every $x_1, x_2, x_3 \in A$ and every $\lambda_1, \lambda_2, \lambda_3 \in B_K$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

We define convex filters in the spirit of [29].

2.3.3 Definition Let A be a B_K-module. A *convex filter* on A is a filter on A that has a base consisting of convex sets. Such a base is called a *convex base* for the filter.

Recall that a filter on A is a collection $\mathcal{F}$ consisting of subsets of A with the properties
1) $A \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$,
2) if $X, Y \in \mathcal{F}$ then $X \cap Y \in \mathcal{F}$,
3) if $X \in \mathcal{F}$ and $Y \subset A$ such that $X \subset Y$ then $Y \in \mathcal{F}$.
A sub-collection $\mathcal{B}$ of $\mathcal{F}$ is called a base for $\mathcal{F}$ if for every $X \in \mathcal{F}$ there exists a $B \in \mathcal{B}$ such that $B \subset X$.

The following proposition is not hard to prove.

2.3.4 Proposition *Let A be a B_K-module. Let C be a non-empty collection of convex subsets of A with the finite intersection property. Let*

$$\mathcal{D} = \{\bigcap_{i=1}^{n} C_i \mid n \in \mathbb{N}, C_1, \ldots, C_n \in C\}.$$

Then $\mathcal{D}$ is a base for a convex filter $\mathcal{F}$ on A.

$\mathcal{F}$ is called the convex filter generated by C.

2.3.5 Definition Let A be a B_K-module. A filter $\mathcal{M}$ on A is called *a maximal convex filter* if $\mathcal{M}$ is a convex filter on A and every convex filter on A in which $\mathcal{M}$ is contained equals $\mathcal{M}$.

The following proposition can easily be proved by using Zorn's Lemma.

2.3.6 Proposition *Let A be a B_K-module and let $\mathcal{F}$ be a convex filter on A. Then there exists a maximal convex filter $\mathcal{M}$ on A such that $\mathcal{F} \subset \mathcal{M}$.*

The following two propositions can be found in [29], (1.6) and (1.7), for K-vector spaces instead of B_K-modules. The proofs remain valid in our more general situation.

2.3.7 Proposition *Let A be a B_K-module. Let $\mathcal{M}$ be a maximal convex filter on A. Let C be a convex subset of A. Then $C \in \mathcal{M}$ or $C \cap D = \emptyset$ for some $D \in \mathcal{M}$.*

2.3.8 Proposition *Let A be a B_K-module. Let C be a system of convex sets that generates a convex filter $\mathcal{F}$ on A. Suppose that for every convex subset D of A either $D \in C$ or $D \cap C = \emptyset$ for some $C \in C$. Then $\mathcal{F}$ is a maximal convex filter and $C = \{D \in \mathcal{F} \mid D \text{ is convex}\}$.*

2.3.9 Corollary *Let A be a B_K-module and let $x \in A$. Let $\mathcal{F}(x)$ be the collection of all subsets of A such that $x \in A$. Then $\mathcal{F}(x)$ is a maximal convex filter on A.*

Linearly Compact B_K-modules

2.3.10 Definition A B_K-module A is called *linearly compact* if for every collection C of convex subsets of A with the finite intersection property we have that $\bigcap C \neq \emptyset$.

2.3.11 Remark For non-spherically complete K there exist B_K-modules $A \neq \{0\}$ such that A is linearly compact.
We will prove that B_K/B_K^- is linearly compact, even if K is not spherically complete. To this end let $(C_\iota)_{\iota\in I}$ be a collection of non-empty convex subsets of B_K/B_K^-. Let, for every $i \in I$, B_ι be the subset of B_K defined by

$$B_\iota = \{\lambda \in B_K \mid \lambda + B_K^- \in C_\iota\}.$$

It is not hard to verify that each B_ι is convex. Hence, for every $i \in I$ there exists a $\lambda_\iota \in B_K$ and an $r_\iota \geq 0$ such that $B_\iota = \mathrm{B}(\lambda_\iota, r_\iota)$. (If $r_\iota = 0$ then $\mathrm{B}(\lambda_\iota, r_\iota) := \{\lambda_\iota\}$. Suppose $r_\iota < 1$ for some $i \in I$. Then $C_\iota = \{\lambda_\iota + B_K^-\}$. (In fact, let $\mu + B_K^- \in C_\iota$. Then $\mu \in \mathrm{B}(\lambda_\iota, r_\iota)$ and hence $|\mu - \lambda_\iota| \leq r_\iota < 1$. Thus, $\mu - \lambda_\iota \in B_K^-$, which implies that $\mu + B_K^- = \lambda_\iota + B_K^-$.) As $(C_\iota)_{\iota\in I}$ has the finite intersection property we obtain that $\lambda_\iota + B_K^- \in \bigcap_{j\in I} C_j$.
If $r_\iota = 1$ for all $i \in I$, then $C_\iota = B_K/B_K^- \quad (i \in I)$ and hence $\bigcap_{\iota\in I} C_\iota = B_K/B_K^-$.

A torsion free B_K-module $A \neq \{0\}$ over a non-spherically complete K can never be linearly compact. This is a well known fact from K-vector space theory.

We will concentrate only on linearly compact modules over spherically complete K.

2.3.12 Proposition *Let A be a B_K-module. Then*
A is linearly compact $\iff$ For every maximal convex filter $\mathcal{M}$ on A there exists an $x \in A$ such that $\mathcal{M} = \mathcal{F}(x)$.

Proof: $\Rightarrow$) Let $\mathcal{M}$ be a maximal convex filter on A. $\mathcal{M}$ has a base, say $(C_\iota)_{\iota\in I}$, consisting of convex subsets of A. Now $(C_\iota)_{\iota\in I}$ has the finite intersection property. Since A is linearly compact there exists an $x \in A$ such that $x \in \bigcap_{\iota\in I} C_\iota$. Then $\{x\} \cap C_\iota \neq \emptyset$ for every $i \in I$ and hence, by Proposition 2.3.7, $\{x\} \in \mathcal{M}$. This implies that $\mathcal{M} = \mathcal{F}(x)$.
$\Leftarrow$) Let $(C_\iota)_{\iota\in I}$ be a collection of convex subsets of A with the finite intersection property. Let $\mathcal{G}$ be the convex filter on A generated by $(C_\iota)_{\iota\in I}$. By Proposition 2.3.6 there exists a maximal convex filter $\mathcal{M}$ on A such that $\mathcal{G} \subset \mathcal{M}$. Let $x \in A$ be such that $\mathcal{M} = \mathcal{F}(x)$. Then $x \in C_\iota$ for every $i \in I$ and hence $\bigcap_{\iota\in I} C_\iota \neq \emptyset$. $\square$

2.3.13 Proposition *Let K be spherically complete. Then*

1) *K is linearly compact.*

2) *Finite direct sums of linearly compact modules are linearly compact.*

3) *Submodules of linearly compact modules are linearly compact.*

4) *A homomorphic image of a linearly compact module is linearly compact.*

Proof: 1) Linear compactness of K means the same as spherical completeness.
2) Let $n \in \mathbb{N}$ and let $A_1, \dots, A_n$ be linearly compact B_K-modules. Let $\mathcal{M}$ be a maximal convex filter on $\bigoplus_{i=1}^n A_i$. Let $\mathcal{B}$ be a convex base for $\mathcal{M}$. For every $j \in \{1, \dots, n\}$ let $P_j : \bigoplus_{i=1}^n A_i \to A_j$ be the projection map. Then the collection $(P_j(B))_{B \in \mathcal{B}}$ consists of convex subsets of A_j and has the finite intersection property. Hence, $\bigcap_{B \in \mathcal{B}} P_j(B) \neq \emptyset$. For every $j \in \{1, \dots, n\}$ let $x_j \in \bigcap_{B \in \mathcal{B}} P_j(B)$. Let $x = (x_1, \dots, x_n) \in \bigoplus_{i=1}^n A_i$. Let $j \in \{1, \dots, n\}$. Then $P_j^{-1}(\{x_j\})$ is a non-empty convex subset of $\bigoplus_{i=1}^n A_i$ and $P_j^{-1}(\{x_j\}) \cap X \neq \emptyset$ for every $X \in \mathcal{B}$. Since $\mathcal{M}$ is a maximal convex filter on A we obtain that $P_j^{-1}(\{x_j\}) \in \mathcal{M}$. Hence $\{x\} = \bigcap_{i=1}^n P_i^{-1}(\{x_i\}) \in \mathcal{M}$, thus $\mathcal{M} = \mathcal{F}(x)$.
By Proposition 2.3.12, $\bigoplus_{i=1}^n A_i$ is linearly compact.
3) Obvious.
4) Let A be a linearly compact B_K-module. Let B be a B_K-module and let $\varphi : A \twoheadrightarrow B$ be a surjective homomorphism. Let $(C_i)_{i \in I}$ be a collection of non-empty convex subsets of B with the finite intersection property. Then $(\varphi^{-1}(C_i))_{i \in I}$ is a collection of non-empty convex subsets of A and it is standard to verify that it has the finite intersection property. Hence, $\bigcap_{i \in I} \varphi^{-1}(C_i) \neq \emptyset$. Thus $\bigcap_{i \in I} C_i = \varphi(\bigcap_{i \in I} \varphi^{-1}(C_i)) \neq \emptyset$. □

2.3.14 Corollary *Let K be spherically complete. Then every B_K-module in $\mathcal{F}_K$ is linearly compact.*

Proof: Combine Proposition 2.2.26 and Proposition 2.3.13. □

We now start proving the converse (see Theorem 2.3.22).

2.3.15 Proposition *Let $A \neq \{0\}$ be a B_K-module such that every finitely generated submodule $B \neq \{0\}$ of A is 1-generated. Then $A \in \mathcal{F}_K$ and* $\operatorname{rank} A = 1$.

Proof: Let $x \in A$ with $x \neq 0$. Let $V = \{\lambda \in K \mid \lambda\{x\} \neq \emptyset\}$ (see 2.1.16 for the definition of $\lambda\{x\}$). Then V is absolutely convex and $B_K \subset V$. We distinguish two cases.
case 1. Suppose that $\sup|V| \in |V|$. Let $\mu \in V$ such that $|\mu| = \sup|V|$ and let $y \in A$ such that $\mu^{-1}y = x$. We prove $A = \operatorname{co}\{y\}$. (Then $\operatorname{rank} A = 1$.)
Let $u \in A$. Then $\operatorname{co}\{u, x\}$ is 1-generated. Then $u \in \operatorname{co}\{x\}$ or $x \in \operatorname{co}\{u\}$. In the first case we obtain $u \in \operatorname{co}\{x\} \subset \operatorname{co}\{y\}$. In the second case there exists a $\nu \in B_K$ such that $\nu u = x$. Then $\nu^{-1} \in V$ and hence $|\nu^{-1}| \leq |\mu|$. Now $u \in \operatorname{co}\{y\}$ or $y \in \operatorname{co}\{u\}$. Suppose $u \notin \operatorname{co}\{y\}$. Then there exists a $\lambda \in B_K^-$ such that $y = \lambda u$. Then

$$(\nu - \mu^{-1}\lambda)u = \nu u - \mu^{-1}(\lambda u) = \nu u - \mu^{-1}y = x - x = 0.$$

Now $|\nu - \mu^{-1}\lambda| = |\nu|$ and

$$(\nu^{-1}(\nu - \mu^{-1}\lambda))x = (\nu^{-1}(\nu - \mu^{-1}\lambda))(\nu u) = (\nu - \mu^{-1}\lambda)u = 0.$$

This is in contradiction with $x \neq 0$. Hence, $u \in \operatorname{co}\{y\}$.
case 2. Suppose that $\sup|V| \notin |V|$. Let $\lambda_0, \lambda_1, \lambda_2, \dots \in K$ be such that

$1 = \lambda_0 < |\lambda_1| < \ldots$ and $\lim_{n\to\infty} |\lambda_n| = \sup |V|$. Then $V = \mathrm{co}\{\lambda_0, \lambda_1, \lambda_2, \ldots\}$. Let $y_0 = x$. Since $\lambda_1 \in V$ there exists a $y_1 \in A$ such that $\lambda_1^{-1} y_1 = y_0$. Let $y_2 \in A$ such that $\frac{\lambda_1}{\lambda_2} y_2 = y_1$. Such an y_2 exists as we may see as follows. There exists a $z \in A$ such that $\lambda_2^{-1} z = x$. Then $\mathrm{rank}\,\mathrm{co}\{y_1, z\} = 1$. In the same way as in case 1. we obtain that $y_1 \in \mathrm{co}\{z\}$. Let $\mu \in B_K$ be such that $y_1 = \mu z$. Then $(\lambda_1^{-1}\mu - \lambda_2^{-1})z = x - x = 0$ and hence, $|\lambda_1^{-1}\mu - \lambda_2^{-1}| < |\lambda_2^{-1}|$ which implies that $|\mu| = |\frac{\lambda_1}{\lambda_2}|$. Let $y_2 = (\frac{\lambda_2}{\lambda_1}\mu)z$. Then $\frac{\lambda_1}{\lambda_2} y_2 = y_1$.
Continuing this way we obtain elements $y_0, y_1, y_2, \ldots$ of A such that $\frac{\lambda_n}{\lambda_{n+1}} y_{n+1} = y_n \quad (n \in \mathbb{N})$. We prove that $A = \mathrm{co}\{y_0, y_1, y_2, \ldots\}$.
Let $u \in A$. Then $\mathrm{rank}\,\mathrm{co}\{y_0, u\} = 1$. If $u \in \mathrm{co}\{y_0\}$ then we are done. If not $u \in \mathrm{co}\{y_0\}$ then there exists a $\mu \in B_K^-$ such that $y_0 = \mu u$. Then $\mu^{-1} \in V$ and hence there exists an $n \in \mathbb{N}$ such that $|\mu^{-1}| < |\lambda_n|$. As we have already seen before this implies that $u \in \mathrm{co}\{y_n\}$ and again we are done.
We define a map $\varphi : \{\lambda_0, \lambda_1, \lambda_2, \ldots\} \to A$ by $\varphi(\lambda_n) = y_n \quad (n \in \mathbb{N})$. We prove by induction that for all $n \in \mathbb{N}$: If $\mu_0, \ldots, \mu_n \in B_K$ such that $\mu_0\lambda_0 + \cdots + \mu_n\lambda_n = 0$ then $\mu_0 y_0 + \cdots + \mu_n y_n = 0$.
For $n = 0$ this is clear. To prove the induction step from $n-1$ to n, let $n \geq 1$ and let $\mu_0, \ldots, \mu_n \in B_K$ such that $\mu_0\lambda_0 + \cdots + \mu_n\lambda_n = 0$. Then $|\mu_n\lambda_n| \leq |\lambda_{n-1}|$. Hence, $\mu_n\lambda_n\lambda_{n-1}^{-1} \in B_K$ and

$$\mu_0\lambda_0 + \cdots + \mu_{n-2}\lambda_{n-2} + (\mu_{n-1} + \mu_n\lambda_n\lambda_{n-1}^{-1})\lambda_{n-1} = 0.$$

Thus,

$$\mu_0 y_0 + \cdots + \mu_n y_n = \mu_0 y_0 + \cdots + \mu_{n-1} y_{n-1} + (\mu_n\lambda_n\lambda_{n-1}^{-1})(\frac{\lambda_{n-1}}{\lambda_n}) y_n =$$
$$\mu_0 y_0 + \cdots + \mu_{n-2} y_{n-2} + (\mu_{n-1} + \mu_n\lambda_n\lambda_{n-1}^{-1}) y_{n-1} = 0$$

(by the induction hypothesis). By Theorem 2.1.29 there exists a homomorphism $\tilde{\varphi} : V \to A$ extending φ. As $A = \mathrm{co}\{y_0, y_1, y_2, \ldots\}$ we obtain that $\tilde{\varphi}$ is surjective. Hence, $A \in \mathcal{F}_K$ and $\mathrm{rank}\, A = 1$. □

2.3.16 Definition Let A be a B_K-module. A submodule B of A is called a *maximal rank 1 submodule* if $\mathrm{rank}\, B = 1$ and for all rank 1 submodules C of A: if $B \subset C$ then $B = C$.

2.3.17 Proposition *Let A be a B_K-module and $x \in A$, $x \neq 0$. Then there exists a maximal rank 1 submodule B of A with $x \in B$.*

Proof: Apply Zorn's Lemma to the class $(C, \supset)$, where C consists of all rank 1 submodules B of A with $x \in B$. (Use Proposition 2.3.15 to show that for every chain $(B_i)_{i\in I}$ the module $\bigcup_{i\in I} B_i$ is an upper bound.) □

2.3.18 Proposition *Let A be a linearly compact B_K-module. Let B be a B_K-module and let $\varphi : A \to B$ be a surjective homomorphism. Let C be an absolutely convex subset of K and let $\rho : C \to B$ be a homomorphism. Then* *there exists a homomorphism $\vartheta : C \to A$ such that the following diagram commutes.*

$$\begin{tikzcd} A \arrow[rr, "\varphi"] & & B \\ & C \arrow[ul, "\vartheta"] \arrow[ur, "\rho"'] & \end{tikzcd}$$

Proof: case 1. There exists a $\mu \in K$ such that $C = \operatorname{co}\{\mu\}$. Let $x \in A$ such that $\varphi(x) = \rho(\mu)$. We define $\vartheta : C \to A$ by $\vartheta(\lambda) = \frac{\lambda}{\mu}x \quad (\lambda \in C)$. Obviously ϑ satisfies the requirements.

case 2. There does not exist a $\mu \in K$ such that $C = \operatorname{co}\{\mu\}$. Then there exist $\lambda_1, \lambda_2, \lambda_3, \ldots \in K$ with $|\lambda_1| < |\lambda_2| < \ldots$ such that $C = \operatorname{co}\{\lambda_1, \lambda_2, \lambda_3, \ldots\}$.
For every $n \geq 1$ the set $\frac{\lambda_1}{\lambda_n}\varphi^{-1}(\rho(\lambda_n))$ is non-empty and convex.

Let $n, m \in \mathbb{N}$ with $m > n$. Let $z \in \frac{\lambda_1}{\lambda_m}\varphi^{-1}(\rho(\lambda_m))$. Let $w \in \varphi^{-1}(\rho(\lambda_m))$ be such that $z = \frac{\lambda_1}{\lambda_m}w$. Then $\varphi(\frac{\lambda_n}{\lambda_m}w) = \frac{\lambda_n}{\lambda_m}\varphi(w) = \frac{\lambda_n}{\lambda_m}\rho(\lambda_m) = \rho(\lambda_n)$. Hence $z = \frac{\lambda_1}{\lambda_n}(\frac{\lambda_n}{\lambda_m}w) \in \frac{\lambda_1}{\lambda_n}\varphi^{-1}(\rho(\lambda_n))$.

We see that $\frac{\lambda_1}{\lambda_m}\varphi^{-1}(\rho(\lambda_m)) \subset \frac{\lambda_1}{\lambda_n}\varphi^{-1}(\rho(\lambda_n))$. Thus, $\{\frac{\lambda_1}{\lambda_n}\varphi^{-1}(\rho(\lambda_n))\}_{n\geq 1}$ is a collection of non-empty convex subsets of A with the finite intersection property. As A is linearly compact we obtain that $\bigcap_{n\geq 1}\frac{\lambda_1}{\lambda_n}\varphi^{-1}(\rho(\lambda_n)) \neq \emptyset$.
Let $z_1 \in \bigcap_{n\geq 1}\frac{\lambda_1}{\lambda_n}\varphi^{-1}(\rho(\lambda_n))$.

In the same way we see that $\{\frac{\lambda_2}{\lambda_n}\varphi^{-1}(\rho(\lambda_n))\}_{n\geq 2}$ is a non-empty collection of convex subsets of A with the finite intersection property.

Let $z_2 \in \bigcap_{n\geq 2}\frac{\lambda_2}{\lambda_n}\varphi^{-1}(\rho(\lambda_n))$ such that $\frac{\lambda_1}{\lambda_2}z_2 = z_1$. We can choose z_2 in such a way since

$$\frac{\lambda_1}{\lambda_2}\bigcap_{n\geq 2}\frac{\lambda_2}{\lambda_n}\varphi^{-1}(\rho(\lambda_n)) = \bigcap_{n\geq 1}\frac{\lambda_1}{\lambda_n}\varphi^{-1}(\rho(\lambda_n)).$$

Continuing this way we find $z_1, z_2, z_3, \ldots \in A$ such that $z_n \in \varphi^{-1}(\rho(\lambda_n))$ and $\frac{\lambda_n}{\lambda_{n+1}}z_{n+1} = z_n$ for all $n \geq 1$.

Let $\hat{\vartheta} : \{\lambda_1, \lambda_2, \lambda_3, \ldots\} \to A$ be defined by $\hat{\vartheta}(\lambda_n) = z_n \quad (n \geq 1)$.

In the same way as in the proof of Proposition 2.3.15 we obtain for every $n \in \mathbb{N}$: if $\mu_1, \ldots, \mu_n \in B_K$ such that $\mu_1\lambda_1 + \cdots + \mu_n\lambda_n = 0$ then $\mu_1 z_1 + \cdots + \mu_n z_n = 0$.

By Theorem 2.1.29 there exists a homomorphism $\vartheta : C \to A$ such that $\vartheta(\lambda_n) = z_n \quad (n \in \mathbb{N})$.

Let $\mu \in C$. Then there exists an $n \geq 1$ such that $|\mu| \leq |\lambda_n|$. Then $\varphi \circ \vartheta(\mu) = \varphi(\vartheta(\frac{\mu}{\lambda_n}\lambda_n)) = \varphi(\frac{\mu}{\lambda_n}\vartheta(\lambda_n)) = \frac{\mu}{\lambda_n}\varphi(z_n) = \frac{\mu}{\lambda_n}\rho(\lambda_n) = \rho(\mu)$.
We see that ϑ fulfills the requirements. □

2.3.19 Remark In the above proposition we can take for C any absolutely convex subset of some K^n.

For let A be linearly compact and let φ be a homomorphism from A to a

B_K-module B. Let C be an absolutely convex subset of K^n. Let $\rho : C \to B$ be a homomorphism. From (iii) of Proposition 2.2.37 we obtain that there exist $k \le n$ absolutely convex subsets $F_1, \dots, F_k$ of K and $x_1, \dots, x_k \in K^n$ such that $C = F_1 x_1 \oplus \cdots \oplus F_k x_k$. Let the map $\rho_i : F_i \to B$ be defined by $\rho_i(\lambda) = \rho(\lambda x_i) \quad (\lambda \in F_i) \quad (i \in \{1, \dots, k\})$. For every $i \in \{1, \dots, k\}$ there exists a map $\tilde{\rho}_i : F_i \to A$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[rr, "\varphi"] & & B \\ & F_i \arrow[ul, "\tilde{\rho}_i"] \arrow[ur, "\rho_i"'] & \end{tikzcd}$$

Then $\tilde{\rho} : C \to A$ defined by

$$\tilde{\rho}(\lambda_1 x_1 + \cdots + \lambda_k x_k) = \tilde{\rho}_1(\lambda_1) + \cdots + \tilde{\rho}_k(\lambda_k) \quad (\lambda_1 \in F_1, \dots, \lambda_k \in F_k)$$

is a homomorphism such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[rr, "\varphi"] & & A \\ & C \arrow[ul, "\tilde{\rho}"] \arrow[ur, "\rho"'] & \end{tikzcd}$$

2.3.20 Proposition *Let K be spherically complete. Let A be a finitely generated B_K-module. Let S be a maximal rank 1 submodule of A. Then S is 1-generated.*

Proof: Let $x_1, \dots, x_n$ be a minimally generating collection for A. Let $e_1, \dots, e_n$ be the canonical base for K^n. Let $\varphi : \mathrm{co}\{e_1, \dots, e_n\} \to A$ be defined by $\varphi(\lambda_1 e_1 + \cdots + \lambda_n e_n) = \lambda_1 x_1 + \cdots + \lambda_n x_n \quad (\lambda_1, \dots, \lambda_n \in B_K)$. Let C be an absolutely convex subset of K and let $\rho : C \to S$ be a surjective homomorphism. By using Proposition 2.3.18 we obtain that there exists a homomorphism $\tilde{\rho} : C \to \mathrm{co}\{e_1, \dots, e_n\}$ such that the following diagram commutes.

$$\begin{tikzcd} \mathrm{co}\{e_1, \dots, e_n\} \arrow[rr, "\varphi"] & & A \\ & C \arrow[ul, "\tilde{\rho}"] \arrow[ur, "\rho"'] & \end{tikzcd}$$

Let $D = [\tilde{\rho}(C)] \cap \mathrm{co}\{e_1, \dots, e_n\}$. Then $\operatorname{rank} D = 1$ and there exist an $i \in \{1, \dots, n\}$ and $\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n \in B_K$ such that

$$y := \lambda_1 e_1 + \cdots + \lambda_{i-1} e_{i-1} + e_i + \lambda_{i+1} e_{i+1} + \cdots + \lambda_n e_n \in D.$$

Now $\varphi(D)$ is a rank 1 submodule of A and $S \subset \varphi(D)$. Hence, $S = \varphi(D)$. We prove $S = \mathrm{co}\{\varphi(y)\}$. To this end let $u \in S$. Then $\operatorname{rank} \mathrm{co}\{\varphi(y), u\} = 1$ and hence there exists a $\lambda \in B_K$ such that $u = \lambda\varphi(y)$ or there exists a $\mu \in B_K$ such that $\varphi(y) = \mu u$. In the first case we obtain that $u \in \mathrm{co}\{\varphi(y)\}$ and we are done. As $\varphi(y) \in A \setminus A^-$ we obtain in second case that $|\mu| = 1$ and hence $u = \mu^{-1}\varphi(y) \in \mathrm{co}\{\varphi(y)\}$. □

2.3.21 Lemma *Let K be spherically complete. Let $A \neq \{0\}$ be a linearly compact B_K-module. Suppose there exists an $n \in \mathbb{N}$ such that every finitely generated submodule of A is m-generated for some $m \leq n$. Then there exist rank 1 submodules $B_1, \ldots, B_n$ of A such that $A = B_1 + \cdots + B_n$.*

Proof: By induction. The case $n = 1$ is Proposition 2.3.15.
Let $n \geq 2$. From Proposition 2.3.17 we obtain that there exists a maximal rank 1 submodule S of A. Let $\pi : A \to A/S$ be the canonical map. We prove that each finitely generated submodule of A/S is m-generated for some $m \leq n-1$. To this end suppose there exists an n-generated submodule B of A/S. Let $x_1, \ldots, x_n \in A$ be such that $B = \mathrm{co}\{\pi(x_1), \ldots, \pi(x_n)\}$. Then $S \subset \mathrm{co}\{x_1, \ldots, x_n\}$. For suppose $s \in S$. Then $\mathrm{co}\{x_1, \ldots, x_n, s\}$ is n-generated. From Corollary 2.2.8 we obtain that $\mathrm{co}\{x_1, \ldots, x_n, s\} = \mathrm{co}\{x_1, \ldots, x_n\}$ or there exists an $i \in \{1, \ldots, n\}$ such that $\mathrm{co}\{x_1, \ldots, x_n, s\} = \mathrm{co}\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, s\}$. The latter can not occur since $\mathrm{co}\{\pi(x_1), \ldots, \pi(x_n)\}$ is n-generated. Thus $\mathrm{co}\{x_1, \ldots, x_n, s\} = \mathrm{co}\{x_1, \ldots, x_n\}$ and hence $s \in \mathrm{co}\{x_1, \ldots, x_n\}$.
As S is a maximal rank 1 submodule of A it is also a maximal rank 1 submodule of $\mathrm{co}\{x_1, \ldots, x_n\}$ and from Proposition 2.3.20 we obtain that S is 1-generated. Let $y \in S$ such that $S = \mathrm{co}\{y\}$. There exist $\lambda_1, \ldots, \lambda_n \in B_K$ such that $y = \lambda_1 x_1 + \ldots + \lambda_n x_n$. Let $i \in \{1, \ldots, n\}$ such that $|\lambda_i| = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$. By renumeration we may suppose that $i = 1$. Then $\lambda_1^{-1}\lambda_i \in B_K$ for every $i \in \{1, \ldots, n\}$ and $z := x_1 + \lambda_1^{-1}\lambda_2 x_2 + \cdots + \lambda_1^{-1}\lambda_n x_n \in \mathrm{co}\{x_1, \ldots, x_n\}$. Since $\lambda_1 z = y$ we obtain that $\mathrm{co}\{y\} \subset \mathrm{co}\{z\}$. Thus, $\mathrm{co}\{y\} = \mathrm{co}\{z\}$.
Then $\mathrm{co}\{z, x_2, \ldots, x_n\} = \mathrm{co}\{x_1, \ldots, x_n\}$. This implies that
$B = \mathrm{co}\{\pi(z), \pi(x_2), \ldots, \pi(x_n)\} = \mathrm{co}\{\pi(x_2), \ldots, \pi(x_n)\}$, a contradiction.
By the induction hypothesis there exist rank 1 submodules $B_1, \ldots, B_{n-1}$ of A/S such that $A/S = B_1 + \cdots + B_{n-1}$. By using Proposition 2.3.18 and Remark 2.3.19 we find rank 1 submodules $C_1, \ldots, C_n$ of A such that $\pi(C_i) = B_i \quad (i = 1, \ldots, n-1)$. Then $A = C_1 + \cdots + C_{n-1} + S$. □

Now we have enough machinery for:

2.3.22 Theorem *Let K be spherically complete, let A be B_K-module. Then $A \in \mathcal{F}_K \iff A$ is linearly compact.*

Proof: $\Rightarrow$) This is Corollary 2.3.14.
$\Leftarrow$) Suppose there exist $e_1, e_2, e_3, \ldots \in A$ such that $\mathrm{co}\{e_1, \ldots, e_n\}$ is n-generated for every $n \geq 1$. Let $B_0 = \mathrm{co}\{e_1, e_2, \ldots\}$ and

$$B_n = (e_1 + \ldots + e_n) + \mathrm{co}\{e_{n+1}, e_{n+2}, \ldots\} \quad (n \geq 1).$$

Then each B_n is convex and $B_0 \supset B_1 \supset \ldots$. As A is linearly compact we obtain that $\bigcap_{n \in \mathbb{N}} B_n \neq \emptyset$.

Let $x \in \bigcap_{n\in\mathbb{N}} B_n$. Then $x \in B_0 = \operatorname{co}\{e_1, e_2, e_3, \dots\}$ and hence there exist an $m \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_m \in B_K$ such that $x = \lambda_1 e_1 + \cdots + \lambda_m e_m$. Now also $x \in B_{m+1}$ and hence there exist $s \geq m+2$ and $\lambda_{m+2}, \dots, \lambda_s \in B_K$ such that $x = e_1 + \cdots + e_{m+1} + \sum_{i=m+2}^{s} \lambda_i e_i$. Then

$$e_{m+1} = (\lambda_1 - 1)e_1 + \cdots + (\lambda_m - 1)e_m + \sum_{i=m+2}^{s} -\lambda_i e_i$$

and hence $e_{m+1} \in \operatorname{co}\{e_1, \dots, e_m, e_{m+2}, \dots, e_s\}$. This is in conflict to our assumption on A.
We may conclude that there exists an $n \in \mathbb{N}$ such that every finitely generated submodule B of A is m-generated for some $m \leq n$.
From the previous lemma it then follows that there exist rank 1 submodules $B_1, \dots, B_n$ of A such that $A = B_1 + \cdots + B_n$. Now $\bigoplus_{i=1}^{n} B_i \in \mathcal{F}_K$ and there exists an obvious surjective homomorphism $\bigoplus_{i=1}^{n} B_i \to B_1 + \cdots + B_n (= A)$. Hence also $A \in \mathcal{F}_K$. □

2.4 Edged and Edge Complete

Recall from [24] that an absolutely convex subset A of a K-vector space E is called edged (in E) if for every $x \in E$ the set $\{|\lambda| \mid \lambda \in K,\ \lambda x \in A\}$ is closed in $|K|$. If $|K|$ is discrete then every absolutely convex set is edged, so the notion is not interesting.

Therefore, in this section the valuation on K is supposed to be dense.

We now have the following, see also [24].

2.4.1 Proposition *Let E be a K-vector space and let A be an absolutely convex subset of E. Then the following assertions are equivalent.*

(α) *A is edged.*

(β) $\bigcap\{\lambda A \mid \lambda \in K, |\lambda| > 1\} = A$.

(γ) *For every $x \in E$, if $\lambda x \in A$ for every $\lambda \in B_K^-$ then $x \in A$.*

In this section we will be dealing with the concept edged in the more general context of B_K-modules.

Edged B_K-modules

The initial definition of edged is not useful for B_K-modules, so we take one of the equivalent formulations.

2.4.2 Definition Let A be a B_K-module and let B be a submodule of A. Then B is called *edged in A* if $\bigcap\{\lambda B \mid \lambda \in K, |\lambda| > 1\} = B$.
(For the definition of λB see 2.1.16).

2.4.3 Remark A submodule B of A is edged in A if and only if for every $x \in A$ the following holds. If $\lambda x \in B$ for every $\lambda \in B_K^-$ then $x \in B$.
A drawback of this definition is that, contrary to the vector space case, the 'edgedness' of B depends on the module where B is embedded in. For example, B_K^- is edged in B_K^- but B_K^- is not edged in B_K.

2.4.4 Definition Let A be a B_K-module. Let B be a submodule of A. We define $B^e = \bigcap\{\lambda B \mid \lambda \in K, |\lambda| > 1\}$.

2.4.5 Proposition *Let A be a B_K-module and let B and C be submodules of A. Then*

(i) $B \subset B^e$ *and* $B^{ee} = B^e$.

(ii) $(B + C^e)^e = (B + C)^e$.

Proof: The proof of (i) is straightforward. We only prove (ii). From (i) it follows that $(B + C^e)^e \supset (B + C)^e$. We prove $(B + C^e)^e \subset (B + C)^e$. To this end suppose $x \in (B + C^e)^e$. Let $\lambda \in B_K^-$. Let $\mu \in B_K^-$ such that $|\mu| > |\lambda|$. Then $\mu x \in B + C^e$, hence $\lambda x = (\lambda\mu^{-1})(\mu x) \in (\lambda\mu^{-1})(B + C^e) \subset B + C$.
We see that $\lambda x \in B + C$ for every $\lambda \in B_K^-$ and hence $x \in (B + C)^e$. □

2.4.6 Remark The following statement is not true.
Let K be spherically complete. Let A be a linearly compact B_K-module. Let B be a B_K-module and let $\varphi : A \to B$ be a homomorphism. Let C be an edged submodule of A. Then $\varphi(C)$ is edged in B.
For example, Let K be spherically complete, $A = B = C = B_K/B_K^-$ and $\varphi = 0$. Then $\varphi(C) = \{0\}$, which is not edged in B.
This is one of the reasons why we will introduce the notion *edge complete* in Definition 2.4.7. This notion is another generalization of 'edgedness' in vector space theory, one that does not depend on the embedding module.
In Theorem 2.4.10 we will prove the following property of edge completeness.
Let K be spherically complete. Let A be a linearly compact B_K-module. Let B be a B_K-module and let $\varphi : A \to B$ be a homomorphism. If A is edge complete then so is $\varphi(A)$.

Edge Complete B_K-modules

2.4.7 Definition A B_K-module A is called *edge complete* if every homomorphism $\varphi : B_K^- \to A$ can be extended to a homomorphism $\tilde{\varphi} : B_K \to A$.

2.4.8 Remark The map $\tilde{\varphi}$ in the previous definition need not be unique. For example, let K be spherically complete and let e_1, e_2 be the canonical base for K^2. Let $A = \mathrm{co}\{e_1, e_2\}/\{\lambda(e_1 - e_2) \mid \lambda \in B_K\}$. Then A is edge complete (we will not prove this here, it follows from Corollary 2.4.14).
Let $\varphi : B_K^- \to A$ be defined by $\varphi(\lambda) = \lambda e_1 + B_K^-(e_1 - e_2)$ $(\lambda \in B_K^-)$. Then $\rho : B_K \to A$ defined by $\rho(\lambda) = \lambda e_1 + B_K^-(e_1 - e_2)$ $(\lambda \in B_K)$ and $\psi : B_K \to A$ defined by $\psi(\lambda) = \lambda e_2 + B_K^-(e_1 - e_2)$ $(\lambda \in B_K)$ are both extensions of φ and $\rho \neq \psi$.

2.4.9 Proposition *Let E be a K-vector space and let A be an absolutely convex subset of E. Then: A is edged (in E) $\iff$ A is edge complete.*

Proof: $\Rightarrow$) Let $\varphi : B_K^- \to A$ be a homomorphism. Let $\mu \in B_K^- \setminus \{0\}$. Let $y = \mu^{-1}\varphi(\mu) \in E$. Then $\varphi(\lambda) = \lambda y$ for every $\lambda \in B_K^-$. This implies that $\lambda y \in A$ for every $\lambda \in B_K^-$ and thus $y \in A^e = A$. Then $\tilde{\varphi} : B_K \to A$ defined by $\tilde{\varphi}(\lambda) = \lambda y \quad (\lambda \in B_K)$ is an extension of φ.
We see that A is edge complete.
$\Leftarrow$) Let $y \in E$ such that $\lambda y \in A$ for every $\lambda \in B_K^-$. The homomorphism $\varphi : B_K^- \to A$ defined by $\varphi(\lambda) = \lambda y \quad (\lambda \in B_K^-)$ has an extension $\tilde{\varphi} : B_K \to A$. As A is torsion free we obtain that $\tilde{\varphi}(1) = y$ and hence $y \in A$. We see that $A = A^e$ and hence A is edged. □

2.4.10 Theorem *Let K be spherically complete. Let A be a linearly compact and edge complete B_K-module. Let B be a B_K-module and let $\varphi : A \to B$ be a surjective homomorphism. Then B is linearly compact and edge complete.*

Proof: From 4) of Proposition 2.3.13 it follows that B is linear compact. We prove that B is edge complete. To this end let $\psi : B_K^- \to B$ be a homomorphism. From Theorem 2.3.22 it follows that $A, B \in \mathcal{F}_K$ and by Remark 2.3.19 there exists a homomorphism $\rho : B_K^- \to A$ such that $\varphi \circ \rho = \psi$. As A is edge complete there exists a $\tilde{\rho} : B_K \to A$ extending ρ. Then $\varphi \circ \tilde{\rho} : B_K \to B$ extends ψ. □

2.4.11 Proposition *Let K be spherically complete. Let C be a submodule of B_K. Then B_K/C is edge complete.*

Proof: It is obvious that B_K is an edged subset of K and from Proposition 2.4.9 we obtain that B_K is edge complete. From Corollary 2.3.14 we obtain that B_K is linearly compact. The canonical map $\pi : B_K \to B_K/C$ is a surjective homomorphism. By using the previous theorem we obtain that B_K/C is edged complete. □

If K is not spherically complete then there exists a submodule C of B_K such that B_K/C is not edge complete, as we see in the following example.

2.4.12 Example Let K be not spherically complete. Let $a_1, a_2, a_3, \ldots \in B_K$ and $1 \ge r_1 > r_2 > r_3 > \ldots$ such that $\mathrm{B}(a_1, r_1) \supset \mathrm{B}(a_2, r_2) \supset \mathrm{B}(a_3, r_3) \supset \ldots$ and $\bigcap_{n\ge1} \mathrm{B}(a_n, r_n) = \emptyset$. Let $r = \lim_{n\to\infty} r_n$. Let $\lambda_1, \lambda_2, \lambda_3, \ldots \in B_K^-$ be such that $r_1 \ge |\lambda_1|^{-1} r > r_2 \ge |\lambda_2|^{-1} r > r_3 \ge \ldots$. Then $|\lambda_1| < |\lambda_2| < \ldots$ and $\lim_{n\to\infty} |\lambda_n| = 1$. Let $s_n = |\lambda_n|^{-1} r \quad (n \ge 1)$.
Then $\mathrm{B}(a_1, s_1) \supset \mathrm{B}(a_2, s_2) \supset \ldots$ and $\bigcap_{n\ge1} \mathrm{B}(a_n, s_n) \subset \bigcap_{n\ge1} \mathrm{B}(a_n, r_n) = \emptyset$.
Let $C = \mathrm{B}(0, r)$. Then $\mathrm{B}(a_n, s_n) = a_n + \lambda_n^{-1} C$ for each $n \ge 1$.
Let $\varphi : \{\lambda_1, \lambda_2, \lambda_3, \ldots\} \to B_K/C$ be defined by $\varphi(\lambda_n) = \lambda_n a_n + C \quad (n \ge 1)$.
We prove inductively: if $n \in \mathbb{N}$, $\mu_1, \ldots, \mu_n \in B_K$ and $\mu_1\lambda_1 + \cdots + \mu_n\lambda_n = 0$ then $\mu_1\lambda_1 a_1 + \cdots + \mu_n\lambda_n a_n \in C$.
For $n = 1$ this is clear.
Let $n \ge 2$ and suppose $\mu_1, \ldots, \mu_n \in B_K$ such that $\mu_1\lambda_1 + \cdots + \mu_n\lambda_n = 0$.
Then $|\mu_n\lambda_n| \le |\lambda_{n-1}|$ and hence $\mu_n\lambda_n\lambda_{n-1}^{-1} \in B_K$.

Then $\mu_1\lambda_1+\cdots+\mu_{n-2}\lambda_{n-2}+(\mu_{n-1}+\mu_n\lambda_n\lambda_{n-1}^{-1})\lambda_{n-1}=0$. By induction then $\mu_1\lambda_1a_1+\cdots+\mu_{n-2}\lambda_{n-2}a_{n-2}+(\mu_{n-1}+\mu_n\lambda_n\lambda_{n-1}^{-1})\lambda_{n-1}a_{n-1}\in C$. Then

$$\mu_1\lambda_1a_1+\cdots+\mu_n\lambda_na_n=\mu_1\lambda_1a_1+\cdots+\mu_{n-2}\lambda_{n-2}a_{n-2}+ (\mu_{n-1}\lambda_{n-1}+\mu_n\lambda_n\lambda_{n-1}^{-1})\lambda_{n-1}a_{n-1}+\mu_n\lambda_n(a_n-a_{n-1}).$$

Hence, $\sum_{i=1}^n\mu_i\lambda_ia_i\in\mu_n\lambda_n(a_n-a_{n-1})+C\subset\mu_n\lambda_n(\lambda_{n-1}^{-1}C)+C\subset C+C=C$. Thus, if $n\in\mathbb{N}$ and $\mu_1,\ldots,\mu_n\in B_K$ are such that $\mu_1\lambda_1+\cdots+\mu_n\lambda_n=0$ then $\mu_1\varphi(\lambda_1)+\cdots+\mu_n\varphi(\lambda_n)=0$. By using Theorem 2.1.29 we obtain that there exists a homomorphism $\psi:B_K^-\to B_K/C$ with $\psi(\lambda_n)=\lambda_na_n+C\quad(n\geq1)$. Suppose $\tilde\psi:B_K\to B_K/C$ is an extension of ψ. Let $a\in B_K$ be such that $\tilde\psi(1)=a+C$. Then for all $n\geq1$: $\lambda_na_n+C=\tilde\psi(\lambda_n)=\lambda_n\tilde\psi(1)=\lambda_na+C$, hence $\lambda_na-\lambda_na_n\in C$. Thus, $a-a_n\in\lambda_n^{-1}C$, implying that $a\in\mathrm{B}(a_n,s_n)$. This is in contradiction to $\bigcap_{n\geq1}\mathrm{B}(a_n,s_n)=\emptyset$.
We see that B_K/C is not edge complete.

2.4.13 Proposition *Let I be an index set and for every $i\in I$ let A_i be an edge complete B_K-module. Then $A:=\prod_{i\in I}A_i$ is edge complete.*

Proof: Let $\varphi:B_K^-\to A$ be a homomorphism. For every $i\in I$ there exists a homomorphism $\tilde\varphi_i:B_K\to A_i$ such that $\tilde\varphi_i|B_K^-=P_i\circ\varphi$. (Here $P_i:A\to A_i$ is the projection map.) Define $\tilde\varphi:B_K\to A$ by $\tilde\varphi(\lambda)=(\tilde\varphi_i(\lambda))_{i\in I}\quad(\lambda\in B_K)$. Then $\tilde\varphi$ is an extension of φ. □

2.4.14 Corollary *Let K be spherically complete. Then all finitely generated B_K-modules are edge complete.*

Proof: Let A be a finitely generated B_K-module. From Proposition 2.2.21 we obtain that there exist $n\in\mathbb{N}$, $k\leq n$ and absolutely convex subsets $C_{k+1},\ldots,C_n$ of K^n such that $A\sim B_K^k\times B_K/C_{k+1}\times\cdots\times B_K/C_n$. Now combine Proposition 2.4.11 and Proposition 2.4.13. □

2.4.15 Definition A B_K-module A is called *simple* if for every submodule B of A either $B=\{0\}$ or $B=A$.

2.4.16 Proposition *Let A be a non-trivial simple B_K-module. Then A is 1-generated. Let $x\in A$, $x\neq0$. Then $\lambda x=0$ for every $\lambda\in B_K^-$. Moreover, A is edge complete.*

Proof: Let $x\in A$ with $x\neq0$. Then $\{0\}\neq\mathrm{co}\{x\}\subset A$ and hence $\mathrm{co}\{x\}=A$. Let $\lambda\in B_K^-$. Then $|1-\lambda\mu|=1$ and hence $x-\mu\lambda x=(1-\mu\lambda)x\neq0$ for every $\mu\in B_K$. Thus, $x\notin\mathrm{co}\{\lambda x\}$. Now $\mathrm{co}\{\lambda x\}\neq\mathrm{co}\{x\}=A$ and hence $\mathrm{co}\{\lambda x\}=\{0\}$. Thus, $\lambda x=0$.
Moreover, it is not hard to verify that 0 is the only homomorphism $B_K^-\to A$. Hence, A is edge complete. □

2.4.17 Proposition *Let A be an edge complete B_K-module and let B be a simple submodule of A. Then A/B is edge complete.*

Proof: Let $\varphi : B_K^- \to A/B$ be a homomorphism. To show that φ can be extended to a homomorphism $\tilde{\varphi} : B_K \to A/B$ we may suppose $\varphi \neq 0$. Let $\lambda \in B_K^-$ such that $\varphi(\lambda) \neq 0$. Let $\lambda_1, \lambda_2, \lambda_3, \ldots \in B_K^-$ be such that $|\lambda| < |\lambda_1| < |\lambda_2| < \ldots$ and $\lim_{n\to\infty} |\lambda_n| = 1$. Let $x_1, x_2, x_3, \cdots \in A$ be such that $\varphi(\lambda_n) = x_n + B \quad (n \geq 1)$.
We define $\vartheta : \{\lambda_1, \lambda_2, \lambda_3, \ldots\} \to A$ by $\vartheta(\lambda_n) = \frac{\lambda_n}{\lambda_{n+1}} x_{n+1} \quad (n \geq 1)$.
For every $n \geq 1$ it follows that $\frac{\lambda_{n+1}}{\lambda_{n+2}} x_{n+2} - x_{n+1} \in B$ and as B is simple we obtain that $\frac{\lambda_n}{\lambda_{n+1}} (\frac{\lambda_{n+1}}{\lambda_{n+2}} x_{n+2} - x_{n+1}) = 0$. This implies that $\frac{\lambda_n}{\lambda_{n+1}} \vartheta(\lambda_{n+1}) = \vartheta(x_n)$ for every $n \geq 1$. From here one easily deduces: if $n \in \mathbb{N}$ and $\mu_1, \ldots, \mu_n \in B_K$ are such that $\mu_1\lambda_1 + \cdots + \mu_n\lambda_n = 0$, then $\mu_1\varphi(\lambda_1) + \cdots + \mu_n\varphi(\lambda_n) = 0$. By using Theorem 2.1.29 we obtain that ϑ can be extended to a homomorphism $\tilde{\vartheta} : B_K^- \to A$. Since A is edge complete, $\tilde{\vartheta}$ can be extended to a homomorphism $\rho : B_K \to A$. Let $\tilde{\varphi} = \pi \circ \rho$, where $\pi : A \to A/B$ is the canonical map. Then $\tilde{\varphi}$ is an extension of φ. □

In the same way we can prove the following proposition.

2.4.18 Proposition *Let A be a B_K-module and let M be the union of all simple submodules of A. Then M is a submodule of A,*

$$M = \{x \in A \mid \lambda x = 0 \text{ for every } \lambda \in B_K^-\}$$

and M is edge complete. Moreover, if A is edge complete then also A/M is edge complete.

2.4.19 Proposition *Let A be a B_K-module and let M be the union of all simple submodules of A. Then:*
$\mathrm{Hom}(B_K^-, A) \sim \mathrm{Hom}(B_K^-, A/M)$.

Proof: Let $\pi : A \to A/M$ be the canonical map. We define a homomorphism $T : \mathrm{Hom}(B_K^-, A) \to \mathrm{Hom}(B_K^-, A/M)$ by $T(\varphi) = \pi \circ \varphi \quad (\varphi \in \mathrm{Hom}(B_K^-, A))$. Let $\varphi \in \mathrm{Hom}(B_K^-, A)$ be such that $T(\varphi) = 0$. Then $\pi \circ \varphi = 0$ and hence $\varphi : B_K^- \to M$, implying $\varphi = 0$. Hence, T is injective.
In the same way as in the proof of Proposition 2.4.17 one shows that for each $\varphi \in \mathrm{Hom}(B_K^-, A/M)$ there exists a $\vartheta \in \mathrm{Hom}(B_K^-, A)$ such that $\pi \circ \vartheta = \varphi$. Hence, T is surjective. □

We denote the class of all B_K-modules that contain no non-trivial simple submodules by $\mathcal{M}$. The following proposition we will use frequently.

2.4.20 Proposition *Let A be a B_K-module. Then:*

$$A \in \mathcal{M} \iff \forall_{x,y \in A}[\forall_{\lambda \in B_K^-}[\lambda x = \lambda y] \Rightarrow x = y].$$

Proof: Straightforward. □

Next we study the connection between the two notions of 'edged' as defined in Definition 2.4.2 and Definition 2.4.7.

2.4.21 Proposition *Let A be an edge complete B_K-module and B a submodule of A that is edged in A. Then B is edge complete.*

Proof: Let $\varphi : B_K^- \to B$ be a homomorphism. Then there exists a homomorphism $\tilde{\varphi} : B_K \to A$ that is an extension of φ. For all $\lambda \in B_K^-$ we have that $\lambda\tilde{\varphi}(1) = \tilde{\varphi}(\lambda) = \varphi(\lambda) \in B$. As B is edged in A we obtain that $\tilde{\varphi}(1) \in B$ and hence $\tilde{\varphi} : B_K \to B$. □

2.4.22 Remark If A is an edge complete B_K-module and B a submodule of A such that B is edge complete, then B is not necessarily edged in A.
For example, take $A = B_K/B_K^-$ and $B = \{0\}$.

2.4.23 Proposition *Let $A \in \mathcal{M}$ be edge complete. Let B be a submodule of A. Then: B is edge complete $\iff$ B is edged in A.*

Proof: By Proposition 2.4.21 we only have to prove $\Rightarrow$). To this end let $x \in A$ be such that $\lambda x \in B$ for every $\lambda \in B_K^-$. Define $\varphi : B_K^- \to B$ by $\varphi(\lambda) = \lambda x$. As B is edge complete there exists a homomorphism $\tilde{\varphi} : B_K \to B$ extending φ. For every $\lambda \in B_K^-$ we have that $\lambda x = \varphi(\lambda) = \tilde{\varphi}(\lambda) = \lambda\tilde{\varphi}(1)$. From Proposition 2.4.20 it follows that $x = \tilde{\varphi}(1) \in B$. □

2.4.24 Proposition *Let A be a B_K-module and let B be a submodule of A. Then: B is edged in A $\iff$ $A/B \in \mathcal{M}$.*

Proof: $\Rightarrow$) Let $y_1, y_2 \in A/B$ such that $\lambda y_1 = \lambda y_2$ for every $\lambda \in B_K^-$. Let $x_1, x_2 \in A$ such that $y_1 = x_1 + B$ and $y_2 = x_2 + B$. Then $\lambda(x_1 - x_2) \in B$ for every $\lambda \in B_K^-$ and hence $x_1 - x_2 \in B$. This implies $y_1 = y_2$.
$\Leftarrow$) Let $x \in A$ such that $\lambda x \in B$ for every $\lambda \in B_K^-$. Then $\lambda x + B = 0$ for every $\lambda \in B_K^-$. As $A/B \in \mathcal{M}$ we obtain that $x + B = 0$ and hence $x \in B$. □

2.4.25 Proposition *Let A be a B_K-module and let M be the union of all simple submodules of A. Then M is edged in A and $A/M \in \mathcal{M}$.*

Proof: Let $x \in A$ such that $\lambda x \in M$ for every $\lambda \in B_K^-$. Let $\mu \in B_K^-$. There exist $\nu, \lambda \in B_K^-$ such that $\mu = \nu\lambda$ (recall that the valuation on K is supposed to be dense). Then $\mu x = \nu(\lambda x) = 0$ since $\lambda x \in M$.
We see that $\mu x = 0$ for every $\mu \in B_K^-$ and hence $x \in M$.
We see that M is edged in A. By using Proposition 2.4.24 we obtain that $A/M \in \mathcal{M}$. □

We now show the existence of an 'edge completion' and discuss its uniqueness.

2.4.26 Theorem *Let $A \in \mathcal{M}$. Then there exist an edge complete B_K-module $A^{\mathrm{edge}} \in \mathcal{M}$ and an injective homomorphism $i : A \to A^{\mathrm{edge}}$ such that for every injective homomorphism T from A to an edge complete B_K-module B there* *exists exactly one injective homomorphism $T' : A^{\text{edge}} \to B$ such that the following diagram commutes.*

$$\begin{tikzcd} A \arrow[r, "T"] \arrow[d, "i"'] & B \\ A^{\text{edge}} \arrow[ur, "T'"'] & \end{tikzcd}$$

Furthermore, if $C \in \mathcal{M}$ is an edge complete B_K-module and j an injective homomorphism $A \to C$ such that for every injective homomorphism T from A to an edge complete module B there exists exactly one injective homomorphism $T' : C \to B$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "T"] \arrow[d, "j"'] & B \\ C \arrow[ur, "T'"'] & \end{tikzcd}$$

commutes, then there exists a unique bijective homomorphism $\Gamma : A^{\text{edge}} \to C$ such that $\Gamma(i(x)) = j(x)$ for every $x \in A$.

Proof: 1. The construction of A^{edge} and $i : A \to A^{\text{edge}}$.
Let $A^{\text{edge}} = \text{Hom}\,(B_K^-, A)$.
Let $\varphi, \vartheta \in A^{\text{edge}}$ such that $\lambda\varphi = \lambda\vartheta$ for every $\lambda \in B_K^-$. Let $\mu \in B_K^-$. There exist $\nu_1, \nu_2 \in B_K^-$ such that $\mu = \nu_1\nu_2$. Then $\varphi(\mu) = \varphi(\nu_1\nu_2) = \nu_1\varphi(\nu_2) = (\nu_1\varphi)(\nu_2) = (\nu_1\vartheta)(\nu_2) = \vartheta(\mu)$. Hence, $\varphi = \vartheta$.
From Proposition 2.4.20 it follows that $A^{\text{edge}} \in \mathcal{M}$.
Let $x \in A$. We define $\varphi_x : B_K^- \to A$ by $\varphi(\lambda) = \lambda x \quad (\lambda \in B_K^-)$.
For $x, y \in A$ and $\lambda \in B_K$ we have that $\varphi_{x+y} = \varphi_x + \varphi_y$ and $\varphi_{\lambda x} = \lambda\varphi_x$.
Hence, $i : A \to A^{\text{edge}}$ defined by $i(x) = \varphi_x$ is a homomorphism $A \to A^{\text{edge}}$.
Now i is injective. In fact, suppose $x, y \in A$ such that $\varphi_x = \varphi_y$. Then $\lambda x = \lambda y$ for every $\lambda \in B_K^-$ and as $A \in \mathcal{M}$ this implies $x = y$.
2. A^{edge} is edge complete.
Let $\lambda \mapsto \varphi_\lambda : B_K^- \to A^{\text{edge}}$ be a homomorphism. We define $\varphi_1 : B_K^- \to A$ as follows.
Let $\lambda \in B_K^-$. Choose $\mu, \nu \in B_K^-$ such that $\lambda = \mu\nu$. Then set $\varphi_1(\lambda) := \varphi_\mu(\nu)$.
This is a well-defined map for suppose also $\mu', \nu' \in B_K$ such that $\lambda = \mu'\nu'$.
By symmetry we may assume that $|\mu| \geq |\mu'|$. Then

$$\varphi_{\mu'}(\nu') = \varphi_{\frac{\mu'}{\mu}\mu}(\nu') = \frac{\mu'}{\mu}\varphi_\mu(\nu') = \varphi_\mu(\frac{\mu'}{\mu}\nu') = \varphi_\mu(\frac{\lambda}{\mu}) = \varphi_\mu(\nu).$$

Now we show that φ_1 is a homomorphism.
Let $\lambda_1, \lambda_2 \in B_K^-$. Let $\mu, \nu_1, \nu_2 \in B_K^-$ such that $\lambda_1 = \mu\nu_1$ and $\lambda_2 = \mu\nu_2$. Then $\varphi_1(\lambda_1 + \lambda_2) = \varphi_1(\mu(\nu_1 + \nu_2)) = \varphi_\mu(\nu_1 + \nu_2) = \varphi_\mu(\nu_1) + \varphi_\mu(\nu_2) = \varphi_1(\lambda_1) + \varphi_1(\lambda_2)$.

Let $\lambda \in B_K^-$ and $\mu \in B_K$. Let $\nu_1, \nu_2 \in B_K^-$ such that $\lambda = \nu_1\nu_2$. Then $\varphi_1(\mu\lambda) = \varphi_1(\mu\nu_1\nu_2) = \varphi_{\nu_1}(\mu\nu_2) = \mu\varphi_{\nu_1}(\nu_2) = \mu\varphi_1(\lambda)$.
We see that $\varphi_1 \in A^{\mathrm{edge}}$.
We define a homomorphism $B_K \longrightarrow A^{\mathrm{edge}}$ by $\lambda \mapsto \lambda\varphi_1$. This homomorphism is an extension of $\lambda \mapsto \varphi_\lambda : B_K^- \longrightarrow A^{\mathrm{edge}}$ as we can see as follows.
Let $\lambda \in B_K^-$. Then for every $\nu \in B_K^-$ we have that $\lambda\varphi_1(\nu) = \varphi_1(\lambda\nu) = \varphi_\lambda(\nu)$, hence $\lambda\varphi_1 = \varphi_\lambda$.
We conclude that A^{edge} is edge complete.
3. The universal property.
Let $B \in \mathcal{M}$ be an edge complete B_K-module. Let $T : A \longrightarrow B$ be an injective homomorphism. We prove that there exists exactly one injective homomorphism $T' : A^{\mathrm{edge}} \longrightarrow B$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "T"] \arrow[d, "i"'] & B \\ A^{\mathrm{edge}} \arrow[ur, "T'"'] & \end{tikzcd}$$

Let $\varphi \in A^{\mathrm{edge}}$. Then $T \circ \varphi$ is a homomorphism $B_K^- \longrightarrow B$. As B is edge complete there exists a homomorphism $\widetilde{T \circ \varphi} : B_K \longrightarrow B$ that is an extension of $T \circ \varphi$. We define $T' : A^{\mathrm{edge}} \longrightarrow B$ by $T'(\varphi) = \widetilde{T \circ \varphi}(1) \quad (\varphi \in A^{\mathrm{edge}})$.
Let $x \in A$. For every $\lambda \in B_K^-$ we obtain that $\lambda T' \circ i(x) = \lambda T'(\varphi_x) = \lambda\widetilde{T \circ \varphi_x}(1) = \widetilde{T \circ \varphi_x}(\lambda) = T \circ \varphi_x(\lambda) = T(\lambda x) = \lambda T(x)$. As $B \in \mathcal{M}$ we obtain that $T' \circ i(x) = T(x)$.
We see that $T' \circ i = T$. The following shows that T' is a homomorphism.
Let $\varphi, \vartheta \in A^{\mathrm{edge}}$. Then for all $\lambda \in B_K^-$ we have that

$$\lambda T'(\varphi + \vartheta) = \lambda\widetilde{T \circ (\varphi + \vartheta)}(1) = \widetilde{T \circ (\varphi + \vartheta)}(\lambda) = T \circ (\varphi + \vartheta)(\lambda) =$$

$$T(\varphi(\lambda) + \vartheta(\lambda)) = T \circ \varphi(\lambda) + T \circ \vartheta(\lambda) = \widetilde{T \circ \varphi}(\lambda) + \widetilde{T \circ \vartheta}(\lambda) =$$

$$\lambda\widetilde{T \circ \varphi}(1) + \lambda\widetilde{T \circ \vartheta}(1) = \lambda(\widetilde{T \circ \varphi}(1) + \widetilde{T \circ \vartheta}(1)) = \lambda(T'(\varphi) + T'(\vartheta)).$$

As $B \in \mathcal{M}$ we obtain that $T'(\varphi + \vartheta) = T'(\varphi) + T'(\vartheta)$.
In a similar way we find that $T'(\mu\varphi) = \mu T'(\varphi)$ for every $\varphi \in A^{\mathrm{edge}}$ and every $\mu \in B_K$.
Furthermore, T' is injective. For suppose $\varphi \in A^{\mathrm{edge}}$ such that $T'(\varphi) = 0$. Then $\widetilde{T \circ \varphi}(1) = 0$ and hence $\widetilde{T \circ \varphi} = 0$, which implies $T \circ \varphi = 0$. As T is injective we obtain that $\varphi = 0$.
To prove the uniqueness of T', suppose that $S : A^{\mathrm{edge}} \longrightarrow B$ is also an injective homomorphism such that $S \circ i = T$. Then $S(\varphi_x) = T(x) = T'(\varphi_x)$ for every $x \in A$. Let $\vartheta \in A^{\mathrm{edge}}$. For every $\lambda \in B_K^-$ we have that $\lambda\vartheta = \varphi_{\vartheta(\lambda)}$ and hence $\lambda S(\vartheta) = S(\lambda\vartheta) = S(\varphi_{\vartheta(\lambda)}) = T'(\varphi_{\vartheta(\lambda)}) = T'(\lambda\vartheta) = \lambda T'(\vartheta)$. As $B \in \mathcal{M}$ we obtain that $S(\vartheta) = T'(\vartheta)$. We see that $S = T'$.
4. We now prove the uniqueness of A^{edge}. Suppose that also $C \in \mathcal{M}$ is an edge complete B_K-module and j an injective homomorphism $A \longrightarrow C$ such that for every injective homomorphism T from A to an edge complete B_K-module B there exists exactly one injective homomorphism $T' : C \longrightarrow B$ such

that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "T"] \arrow[d, "j"'] & B \\ C \arrow[ur, "T'"'] & \end{tikzcd}$$

There exist exactly one injective homomorphism $T : A^{\text{edge}} \to C$ and exactly one injective homomorphism $S : C \to A^{\text{edge}}$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & A^{\text{edge}} \arrow[dl, bend left, "T"] \\ C \arrow[ur, bend left, "S"] & \end{tikzcd}$$

Then $S \circ T(i(x)) = S(T(i(x))) = S(j(x)) = i(x)$ for every $x \in A$ and hence the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "i"'] & A^{\text{edge}} \\ A^{\text{edge}} \arrow[ur, "S \circ T"'] & \end{tikzcd}$$

This implies that $S \circ T = \mathrm{id}_{A^{\text{edge}}}$ ($\mathrm{id}_{A^{\text{edge}}}$ is the identity map on A^{edge}). In the same way we obtain that $T \circ S = \mathrm{id}_C$ and hence T is a bijective homomorphism.

Thus, by taking $\Gamma = T$, the last part of the theorem is proved. $\square$

Chapter 3

Locally Convex B_K-modules

In this chapter we will introduce the notion of a locally convex B_K-module and we will set up some general theory about locally convex B_K-modules, mainly in relation with seminorms. Seminorms are introduced in section 3.3.

From now on we consider K, equipped with the metric induced by the valuation on K. We always assume that K is complete with respect to this metric.

3.1 Topological B_K-modules

3.1.1 Definition A *topological B_K-module* is a pair (A, τ), where A is a B_K-module and τ a topology on A such that the addition $A \times A \to A$ and the scalar multiplication $B_K \times A \to A$ are continuous maps.

Let (A, τ) be a topological B_K-module. Let $(x_\alpha)_{\alpha \in I}$ be a net in A. We write $x_\alpha \xrightarrow{\tau} x$ to express that the net $(x_\alpha)_{\alpha \in I}$ converges to x. If there is no ambiguity about the topology we also write $x_\alpha \to x$. Let $(X_\alpha)_{\alpha \in I}$ be a net of subsets of A such that for every zero neighbourhood U of A there exists a $\gamma \in I$ such that $X_\alpha \subset U$ for all $\alpha \succ \gamma$ then we denote this by $X_\alpha \xrightarrow{\tau} \{0\}$.
The closure of a subset X of A is denoted $\overline{X}$. We also write $\overline{\text{co}}\, X$ instead of $\overline{\text{co}\, X}$.
Let σ be a topology on A. By $\sigma \leq \tau$ we express the fact that σ is weaker than τ.

The following proposition is not hard to prove.

3.1.2 Proposition *Let (A, τ) be a topological B_K-module. Let B be a submodule of A. Then B, equipped with the relative topology, is a topological B_K-module.*

The relative topology on B induced by τ is also called the *restricted topology* and is denoted $\tau|B$.

We recall the following definition from topological vector space theory.

3.1.3 Definition Let A be a K-vector space. Let τ be a topology on A. Then (E, τ) is called a *topological K-vector space* if the addition $A \times A \to A$ and the scalar multiplication $K \times A \to A$ are continuous.

3.1.4 Examples Every topological K-vector space is also a topological B_K-module. (If $|K|$ is trivial then the notion of a topological B_K-module is the same as that of a topological K-vector space.) Each absolutely convex subset of a topological K-vector space equipped with the restricted topology is a topological B_K-module.

3.1.5 Definition Let (A, τ) and (B, σ) be topological B_K-modules and let $\varphi : A \to B$ be a homomorphism. Then φ is called a *quotient map* if φ is surjective, open and continuous. The map φ is called a *homeomorphism from A in B* if the map $\varphi : (A, \tau) \to (\operatorname{Im} \varphi, \sigma | \operatorname{Im} \varphi)$ is a homeomorphism. (A, τ) is called *topologically embeddable in* (B, σ) if there exists a homeomorphism from (A, τ) into (B, σ).

For a topological B_K-module (A, τ) the addition: $A \times A \to A$ and the multiplication by -1: $A \to A$ are continuous, hence (A, τ) is a topological group. The following proposition is a consequence of this fact. (See [13], (4.2), (4.5) and (4.7))

3.1.6 Proposition *Let (A, τ) be a topological B_K-module.*

1. *For every $y \in A$ the map $T_y : A \to A$ defined by $T_y(x) = y + x \quad (x \in A)$ is a homeomorphism.*

2. *For every zero neighbourhood U there exists a zero neighbourhood V such that $V + V \subset U$.*

3. *(A, τ) has a base of zero neighbourhoods consisting of closed sets.*

4. *Let C be a base of zero neighbourhoods for (A, τ). Then: $V \subset A$ is open $\iff$ For every $x \in V$ there exists a $U \in C$ such that $x + U \subset V$.*

5. *(A, τ) is Hausdorff $\iff$ For every $x \in A$, $x \neq 0$ there exists a zero neighbourhood U such that $x \notin U$.*

6. *Let (B, σ) be a topological B_K-module and let $\varphi : A \to B$ be a homomorphism. Then φ is uniformly continuous $\iff$ φ is continuous $\iff$ φ is continuous at 0. Furthermore, φ is open $\iff$ φ is open at 0 (that is to say for every zero neighbourhood U of A the set $\varphi(U)$ is a zero neighbourhood in B).*

From the continuity of the scalar multiplication $B_K \times A \to A$ we obtain the following. We omit the straightforward proofs.

3.1.7 Proposition *Let (A, τ) be a topological B_K-module.*

1. *If the valuation on K is non-trivial, then every zero neighbourhood in A is absorbing.*
2. *For every zero neighbourhood U and every $\lambda \in B_K \setminus \{0\}$ the set $\lambda^{-1}U$ is a zero neighbourhood.*
3. *For every $\lambda \in B_K$ the map $M_\lambda : A \to A$ defined by $M_\lambda(x) = \lambda x \quad (x \in A)$ is continuous.*
4. *Let B be a submodule of A and $\lambda \in B_K$. Then $\lambda\overline{B} \subset \overline{\lambda B}$.*

Proof: We only prove 1). Let the valuation on K be non-trivial. Let U be a zero neighbourhood in A. Let $x \in A$. Since the scalar multiplication $B_K \times A \to A$ is continuous we obtain that there exists a $\delta > 0$ such that $\lambda x \in U$ for every $\lambda \in B_K$ with $|\lambda| < \delta$. As $|K|$ is non-trivial there exists a $\mu \in B_K$ with $0 < |\mu| < \delta$. Then $\mu x \in U$.
We obtain that U is absorbing. □

3.1.8 Remark If the valuation on K is trivial, then it is not true that every zero neighbourhood in a locally convex B_K-module is absorbing. In fact, let $A \neq \{0\}$. Let d be the discrete topology on A. Then (A, d) is a topological B_K-module and $\{0\}$ is a zero neighbourhood, which is not absorbing.

3.1.9 Remark In a topological B_K-module (A, τ) the map $x \mapsto \lambda x$ is a continuous homomorphism $A \to \lambda A$ for every $\lambda \in B_K^-$. (It is easily seen that if $|\lambda| = 1$ the map $x \mapsto \lambda x$ is a homeomorphism $A \to A$.) But, contrary to the case in vector space theory, this map need not be a homeomorphism. Indeed, if A has torsion elements the map is not injective for all $\lambda \in B_K^-$. In Example 3.2.13 we will see that there exist even torsion free B_K-modules for which the map $x \mapsto \lambda x$: $A \to \lambda A$ is a homeomorphism for no $\lambda \in B_K^-$.

The following simple observation plays a key role in the sequel.

3.1.10 Proposition *Let (A, τ) be a topological B_K-module and let U be a zero neighbourhood that is a submodule of A. Then U is open and closed.*

Proof: We first prove that U is open. To this end observe that for every $x \in U$ we have that $x + U \subset U + U = U$ and hence, by 4. of Proposition 3.1.6, U is open. To show that U is closed, let $x \in A \setminus U$. Then the coset $x + U$ is a neighbourhood of x and $(x + U) \cap U = \emptyset$. We see that $A \setminus U$ is open and hence U is closed. □

3.1.11 Definition A Hausdorff topological B_K-module is called *complete* if every Cauchy net in A converges.

Recall that a *Cauchy net* in a topological B_K-module (A, τ) is a net $(x_\alpha)_{\alpha \in I}$ such that for every zero neighbourhood U of A there exists a $\gamma \in I$ such that $x_\alpha - x_\beta \in U$ for all $\alpha, \beta \succ \gamma$.

The following proposition has its counterpart in vector space theory. The proof is similar.

3.1.12 Proposition *Let (A, τ) be a topological B_K-module and let B be a dense submodule of A. Let (C, σ) be a complete Hausdorff topological B_K-module and $\varphi : B \to C$ a continuous homomorphism. Then there exists precisely one continuous homomorphism $\tilde{\varphi} : A \to C$ such that $\tilde{\varphi}|B = \varphi$.*

3.1.13 Proposition *Let (A, τ) be a topological B_K-module and let B be a submodule of A. Let $\pi : A \to A/B$ be the canonical map. Let*

$$\sigma = \{\pi(U) \mid U \text{ open subset of } A\}.$$

Then $(A/B, \sigma)$ is a topological B_K-module and $\pi : (A, \tau) \to (A/B, \sigma)$ is a quotient map. Furthermore: $(A/B, \sigma)$ is Hausdorff $\iff$ B is closed.

The topology σ defined in the previous proposition is called the *quotient topology.*

The Product Topology

3.1.14 Definition Let $((A_\iota, \tau_\iota))_{\iota\in I}$ be a collection of topological B_K-modules. Let $A = \prod_{\iota\in I} A_\iota$. The *product topology* τ on A is the weakest topology on A such that the projections $P_\iota : (A, \tau) \to (A_\iota, \tau_\iota) \quad (\iota \in I)$ are continuous.

The following proposition has its counterpart in vector space theory. The proof is similar.

3.1.15 Proposition *Let $((A_\iota, \tau_\iota))_{\iota\in I}$ be a collection of topological B_K-modules. Let $A = \prod_{\iota\in I} A_\iota$ and τ the product topology on A. Then (A, τ) is a topological B_K-module.*

1. *If, for every $i \in I$, $\mathcal{F}_\iota$ is a base of zero neighbourhoods for τ_ι then the collection $\{P_\iota^{-1}(U) \mid U \in \mathcal{F}_\iota \quad (i \in I)\}$ is a subbase of zero neighbourhoods for τ.*

2. *If $(x_\alpha)_{\alpha\in J}$ is a net in A and $x \in A$ then*

$$x_\alpha \xrightarrow{\tau} x \text{ in } A \iff x_\alpha(i) \xrightarrow{\tau_\iota} x(i) \text{ in } A_\iota \text{ for every } \iota \in I.$$

3.1.16 Corollary *A product of complete topological B_K-modules is complete.*

The following two propositions are special cases of Theorem 2.5 in chapter IV of [9].

3.1.17 Proposition *Let $((A_\iota, \tau_\iota))_{\iota\in I}$ and $((B_\iota, \sigma_\iota))_{\iota\in I}$ be topological B_K-modules. Suppose that for every $i \in I$ there exists a homeomorphism j_ι from (A_ι, τ_ι) in (B_ι, σ_ι). Let $A = \prod_{\iota\in I} A_\iota$ and let τ be the product topology on A. Let $B = \prod_{\iota\in I} B_\iota$ and let σ be the product topology on B. Let the homomorphism $\Phi : (A, \tau) \to (B, \sigma)$ be defined by*

$$\Phi(x)(i) = j_\iota(x(i)) \quad (i \in I), \quad (x \in \prod_{\iota\in I} A_\iota).$$

Then Φ is a homeomorphism from (A, τ) in (B, τ).

3.1.18 Proposition *Let $((A_i, \tau_i))_{i\in I}$ and $((B_i, \sigma_i))_{i\in I}$ be topological B_K-modules. Suppose that for every $i \in I$ there exists a quotient map $\varphi_i : (A_i, \tau_i) \to (B_i, \sigma_i)$. Let $A = \prod_{i\in I} A_i$ and let τ be the product topology on A. Let $B = \prod_{i\in I} B_i$ and let σ be the product topology on B. Let $\Phi : (A, \tau) \to (B, \sigma)$ be defined by*

$$\Phi(x)(i) = \varphi_i(x(i)) \quad (i \in I), \quad (x \in \prod_{i\in I} A_i).$$

Then Φ is a quotient map.

The following proposition can be proved from 2.7 from chapter IV of [9].

3.1.19 Proposition *Let $((A_n, \tau_n))_{n\in \mathbb{N}}$ be topological B_K-modules that are topologically embeddable in $(K^{\mathbb{N}}, \sigma)$, where σ is the product topology on $K^{\mathbb{N}}$. Let $A = \prod_{n\in\mathbb{N}} A_n$ and let τ be the product topology on A. Then (A, τ) is also topologically embeddable in $(K^{\mathbb{N}}, \sigma)$.*

Locally Convex B_K-modules

The notion of a topological B_K-module is fairly general. To obtain interesting results we will concentrate on the class of the so-called locally convex B_K-modules. In [31] the following definition of a locally convex K-vector space is given.

3.1.20 Definition A topological K-vector space (E, τ) is called a *locally convex K-vector space* if there exists a base of zero neighbourhoods of E consisting of absolutely convex subsets of E.

We translate this definition to a B_K-module version in the following way.

3.1.21 Definition A topological B_K-module (A, τ) is called a *locally convex B_K-module* if τ has a base of zero neighbourhoods consisting of submodules of A. The topology τ is called a *locally convex topology.*

3.1.22 Examples Every locally convex K-vector space is also a locally convex B_K-module and each absolutely convex subset of a locally convex K-vector space, equipped with the restricted topology, is a locally convex B_K-module.

The next Proposition is not hard to verify.

3.1.23 Proposition

(i) *Restricted topologies of locally convex topologies are locally convex (see Proposition 3.1.2).*

(ii) *Quotient topologies of locally convex topologies are locally convex (see Proposition 3.1.13).*

(iii) *The product topology of locally convex topologies is locally convex (see Proposition 3.1.15).*

3.1.24 Definition Let A be a B_K-module. Let C be a non-empty collection of submodules of A such that for every finite subcollection $\mathcal{F}$ of C there exists a $U \in C$ such that $U \subset \bigcap \mathcal{F}$. If $|K|$ is non-trivial then every member of C is supposed to be absorbing (see also 1. of Proposition 3.1.7).
A subset V of A is called *C-open* if for every $y \in V$ there exists a $U \in C$ such that $y + U \subset V$. The collection of all C-open subsets forms a topology that is called the *C-topology.*

The proof of the following proposition is straightforward.

3.1.25 Proposition *Let (A, τ) be a locally convex B_K-module and let $\mathcal{D}$ be a base of zero neighbourhoods of τ consisting of submodules of A. Then for every finite subcollection $\mathcal{F}$ of $\mathcal{D}$ there exists a $U \in \mathcal{D}$ such that $U \subset \bigcap \mathcal{F}$ and, if $|K|$ is non-trivial, every member of $\mathcal{D}$ is absorbing. Moreover, τ is equal to the $\mathcal{D}$-topology.*

3.1.26 Proposition *Let A be a B_K-module and let C be a non-empty collection of submodules of A such that for every finite subcollection $\mathcal{F}$ of C there exists a $U \in C$ such that $U \subset \bigcap \mathcal{F}$. If $|K|$ is non-trivial every member of C is supposed to be absorbing. Let τ be the C-topology. Then (A, τ) is a locally convex B_K-module and C is a base of zero neighbourhoods for τ.*

Proof: We first check that the addition $S : A \times A \to A$ is continuous. Let $x, y \in A$ and let V be a τ-open subset with $x + y \in V$. There exists a $C \in C$ such that $x + y + C \subset V$. Now $z + C = x + C$ for every $z \in x + C$ and hence $x + C$ is τ-open. Likewise $y + C$ is τ-open and

$$S((x + C) \times (y + C)) = (x + C) + (y + C) = (x + y) + C \subset V.$$

We now prove that the scalar multiplication $M : B_K \times A \to A$ is continuous. Suppose $|K|$ is trivial. Let $\lambda \in B_K$ and $x \in A$. Let V be a τ-open subset of A with $\lambda x \in V$. There exists a $U \in C$ such that $\lambda x + U \subset V$. Now $\{\lambda\}$ is open in B_K and $x + U$ is open in A and

$$M(\{\lambda\} \times x + U) = \lambda x + \lambda U \subset \lambda x + U \subset V.$$

We see that M is continuous.
Suppose $|K|$ is non-trivial. Then every member of C is absorbing. Let $\lambda \in B_K$ and $x \in A$. Let V be a τ-open subset such that $\lambda x \in V$. There exists a $C \in C$ such that $\lambda x + C \subset V$. As C is absorbing there exists a $\mu \in B_K \setminus \{0\}$ such that $\mu x \in C$. Let $D = \{\nu \in B_K \mid |\nu - \lambda| \leq |\mu|\}$. Then $D \times (x + C)$ is open in $B_K \times A$ and $(\lambda, x) \in D \times (x + C)$. Let $(\nu, y) \in D \times (x + C)$. Then there exists a $\nu' \in B_K$ such that $\nu = \lambda + \nu'\mu$. Furthermore, $y - x \in C$ and $\nu y = \nu x + \nu(y - x) = (\lambda + \nu'\mu)x + \nu(y - x) = \lambda x + \nu'(\mu x) + \nu(y - x)$. Now $\nu'(\mu x)$, $\nu(y - x) \in C$ and hence $\lambda y = \lambda x + \nu'(\mu x) + \nu(y - x) \in \lambda x + C$. We obtain that $M(D \times (x + C)) \subset \lambda x + C \subset V$.
We see that (A, τ) is a topological B_K-module. It is clear from the definition of τ that C is a base of zero neighbourhoods for τ and C consists of submodules of A. Hence, (A, τ) is locally convex. □

3.1.27 Remark From the previous proposition we obtain that every B_K-module can be equipped with a locally convex topology. The trivial topology on a B_K-module A is locally convex. It is obtained by taking $C = \{A\}$. If $A \neq \{0\}$ this topology is not Hausdorff. If $|K|$ is trivial we can take for C the collection of all submodules of A, if $|K|$ is non-trivial, we can take for C the collection of all absorbing submodules of A. Then the C-topology is the strongest locally convex topology on A. In Lemma 4.1.11 we will prove that it is Hausdorff. (If $|K|$ is trivial, it is the discrete topology).

Not every topological B_K-module is also locally convex as we can see from the following example.

3.1.28 Example Let the valuation on K be non-trivial. Let

$$A = l^1(K) := \{(\lambda_0, \lambda_1, \lambda_2, \dots) \in K^{\mathbb{N}} \mid \sum_{n=0}^{\infty} |\lambda_n| < \infty\}.$$

Then A is a K-vector space. Let $\| \ \|$ on A be defined by

$$\|(\lambda_0, \lambda_1, \lambda_2, \dots)\| = \sum_{n=0}^{\infty} |\lambda_n| \quad ((\lambda_0, \lambda_1, \lambda_2, \dots) \in A).$$

This $\| \ \|$ is an archimedean norm (or a so-called A-norm in the sense of [20], page 88) and the topology induced by it in the standard way is a vector space topology so it certainly makes A into a topological B_K-module. Let $\lambda \in K$ with $0 < |\lambda| < 1$. Let $0 < \varepsilon < |\lambda|$. Let $U = \{x \in A \mid \|x\| < \varepsilon\}$. Let V be a zero neighbourhood such that $V \subset U$. We prove that V is not a submodule of A. In fact, there exists a $\delta > 0$ such that $\{x \in A \mid \|x\| < \delta\} \subset V$. Let $m \in \mathbb{N}$ such that $|\lambda|^m < \delta$ and let $l \in \mathbb{N}$ such that $l|\lambda|^m \geq \varepsilon$. Let $e_0, e_1, e_2, \dots$ be the canonical unit vectors in A. Then $\lambda^m e_0$, $\lambda^m e_1, \dots, \lambda^m e_{l-1} \in V$. But $\|\sum_{i=0}^{l-1} \lambda^m e_i\| = l|\lambda|^m \geq \varepsilon$ and hence $\sum_{i=1}^{l-1} \lambda^m e_i \notin V$. Thus, V is not a submodule.
We see that $(A, \| \ \|)$ is not a locally convex B_K-module.

As a locally convex B_K-module has a base of zero neighbourhoods consisting of submodules we can state the following.

3.1.29 Proposition *Let (A, τ) be a locally convex B_K-module and let $(x_\alpha)_{\alpha \in I}$ be a net in A with $x_\alpha \xrightarrow{\tau} 0$. Let $(\lambda_\alpha)_{\alpha \in I}$ be a net in B_K. Then also $\lambda_\alpha x_\alpha \xrightarrow{\tau} 0$.*

3.1.30 Proposition *Let (A, τ) be a locally convex B_K-module and let B be a submodule of A. Let U be a submodule of B that is open in B with respect to the restricted topology. Then there exists an open submodule V of A such that $U = V \cap B$.*

Proof: There exists an open set W such that $U = W \cap B$. Then $0 \in W$ and there exists an open submodule W' of A such that $W' \subset W$. Let $V = W' + U$. Then V is a submodule of A and V is open since it is a union of additive cosets of W'. Now $U = U \cap B \subset V \cap B$. Furthermore, $V \cap B = (W' + U) \cap B = W' \cap B + U \subset W \cap B + U = U + U = U$. We obtain that $V \cap B = U$. □

3.1.31 Proposition *Let (A, τ) be a Hausdorff locally convex B_K-module and let C be a base of zero neighbourhoods consisting of submodules of A. Then (A, τ) is topologically embeddable in $\prod_{U \in C} A/U$, provided with the product topology. (Here each A/U is provided with the quotient topology which is the discrete topology).*

Proof: We define a homomorphism $\Phi : A \to \prod_{U \in C} A/U$ by

$$\Phi(x)(U) = x + U \quad (U \in C,\ x \in A).$$

Let $x \in A$ be such that $\Phi(x) = 0$. Then $x \in U$ for every $U \in C$. As (A, τ) is Hausdorff we obtain $x = 0$. We see that Φ is injective.
Φ is also continuous. For let $(x_\alpha)_{\alpha \in I}$ be a net in A such that $x_\alpha \xrightarrow{\tau} 0$. Let $U \in C$. Then $x_\alpha \in U$ for large α and hence $\Phi(x_\alpha)(U) = U = 0$ in A/U for large α. By using Proposition 3.1.15 we obtain that $\Phi(x_\alpha) \to 0$ in $\prod_{U \in C} A/U$. Finally, Φ is open. For let $(x_\alpha)_{\alpha \in I}$ be a net in A such that $\Phi(x_\alpha) \to 0$ in $\prod_{U \in C} A/U$. Then for every $U \in C$ we have that $x_\alpha + U = \Phi(x_\alpha)(U) = 0$ for large α and hence $x_\alpha \in U$ for large α. As C is a base of zero neighbourhoods of τ it follows that $x_\alpha \xrightarrow{\tau} 0$. □

3.1.32 Corollary *Each Hausdorff locally convex B_K-module is topologically embeddable in a product of discrete B_K-modules.*

Compare the fact that each Hausdorff locally convex space is a subspace of a product of Banach spaces.

3.1.33 Definition Let (A, τ) be a Hausdorff locally convex B_K-module. A *completion* of (A, τ) is a a complete (Hausdorff) locally convex B_K-module (B, σ) together with a homeomorphism $i : (A, \tau) \to (B, \sigma)$ such that $i(A)$ is dense in B.

3.1.34 Proposition *Let (A, τ) be a Hausdorff locally convex B_K-module and let $((B, \sigma), i)$ and $((B', \sigma'), i')$ be completions of (A, τ). Then there exists a unique homeomorphism $j : (B, \sigma) \to (B', \sigma')$ such that $j(i(x)) = i'(x)$ for every $x \in A$.*

Proof: We consider i as a homeomorphism $A \to i(A)$. Now $i(A)$ is dense in (B, σ) and $i' \circ i^{-1} : i(A) \to B'$ is a continuous homomorphism. From Proposition 3.1.12 we obtain that there exists a unique continuous homomorphism $j : B \to B'$ such that $j | i(A) = i' \circ i^{-1}$. In the same way we obtain that there exists a (unique) continuous homomorphism $j' : B' \to B$ such that $j' | i'(A) = i \circ (i')^{-1}$. Then $j' \circ j(i(x)) = j'(i'(x)) = i(x)$ for all $x \in A$ and hence, again by using Proposition 3.1.12, we obtain that $j' \circ j = \mathrm{id}_B$. In a similar way we obtain $j \circ j' = \mathrm{id}_{B'}$. As j and j' are continuous it follows that $j : (B, \sigma) \to (B', \sigma')$ is a homeomorphism. □

3.1.35 Proposition *Every Hausdorff locally convex B_K-module has a completion.*

Proof: Let (A, τ) be a Hausdorff locally convex B_K-module. Let C be a base of zero neighbourhoods consisting of submodules of A. From Proposition 3.1.31 we obtain that there exists a homeomorphism from (A, τ) in $\prod_{U \in C} A/U$. As each A/U is complete (with respect to the discrete topology) the product $\prod_{U \in C} A/U$ is complete. From Proposition 3.1.12 it follows that $\overline{i(A)}$ is a completion of A. □

Henceforth we will talk about *the* completion of a Hausdorff locally convex module (A, τ) and we will denote this completion $(\hat{A}, \hat{\tau})$ or shortly $\hat{A}$. And we will consider (A, τ) as a submodule of $\hat{A}$.

3.1.36 Remark One can also prove that every Hausdorff topological B_K-module has a completion. But the proof is much more complicated and similar to the one that every Hausdorff topological K-vector space has a completion (see 5. of part I in [32]).

3.2 Normed B_K-modules

3.2.1 Definition A *norm* on a B_K-module A is a map $\|\ \| : A \to [0, \infty)$ with the following properties.

(i) $\|x\| = 0 \iff x = 0 \quad (x \in A)$.

(ii) $\|x + y\| \leq \max(\|x\|, \|y\|) \quad (x, y \in A)$.

(iii) $\|\lambda x\| \leq \|x\| \quad (\lambda \in B_K,\ x \in A)$.
In particular, $\|\lambda x\| = \|x\| \quad (x \in A,\ \lambda \in B_K,\ |\lambda| = 1)$.

(iv) If $x \in A$ and $(\lambda_n)_{n \in \mathbb{N}} \in B_K$ such that $\lambda_n \to 0$, then $\|\lambda_n x\| \to 0$.

If $\|\ \|$ is a norm on A then $(A, \|\ \|)$ is called a *normed B_K-module.*
A norm $\|\ \|$ on A is called a *Bosch norm* (see [5], Definition 1 of section 2.1) if $\|\ \|$ has the properties (i) and (ii) and in addition

(iii)′ $\|\lambda x\| \leq |\lambda| \|x\| \quad (\lambda \in B_K, x \in A)$.

A norm $\|\ \|$ is called *faithful* if, besides (i) and (ii), $\|\ \|$ also fulfills

(iii)″ $\|\lambda x\| = |\lambda| \|x\| \quad (\lambda \in B_K, x \in A)$.

Note that a faithful norm is a Bosch norm and that a Bosch norm satisfies also property (iii) and (iv).

To avoid confusion with norms on vector spaces we define the following.

3.2.2 Definition Let E be a K vector space. A norm $\|\ \|$ on E is called a *vector space norm* if $\|\lambda x\| = |\lambda| \|x\|$ for every $\lambda \in K$ and every $x \in E$.

A *normed K-vector space* is a pair $(E, \|\ \|)$ where E is a K-vector space and $\|\ \|$ is a vector space norm on E.

3.2.3 Remark If A is an absolutely convex subset of a normed K-vector space $(E, \| \ \|)$ then the restriction of $\| \ \|$ to A is a faithful norm on A.

Next we give an example to show that not every norm is a Bosch norm and that a Bosch norm need not be faithful.

3.2.4 Example Let $|K|$ be non-trivial. Let $A = B_K$. Define $\eta : A \to [0, \infty)$ by $\eta(x) = \sqrt{|x|} \quad (x \in A)$. Then η is a norm on A.
But, let $x \in A\setminus\{0\}$. Then $\eta(\lambda x) = \sqrt{|\lambda|}\eta(x) > |\lambda|\eta(x)$ for every $\lambda \in B_K^- \setminus \{0\}$ and hence η is not a Bosch norm.
The map $\nu : A \to [0, \infty)$ defined by $\nu(x) = |x|^2$ is a Bosch norm on A, but it is easily seen that it is not faithful.

3.2.5 Theorem *Let $A \neq \{0\}$ be a B_K-module and let B be a submodule of A with $B \neq \{0\}$. Then any bounded norm $\| \ \|$ on B can be extended to a norm $\| \ \|'$ on A such that* $\sup \| \ \|' = \sup \| \ \|$.

Proof: We consider the collection C of pairs (C, ν) where C is a submodule of A with $B \subset C$ and ν is a norm on C such that $\nu|B = \| \ \|$ and $\sup \nu = \sup \| \ \|$. Then $C \neq \emptyset$ since $(B, \| \ \|) \in C$. Let the partial order $\succ$ on C be defined by $(C', \nu') \succ (C, \nu)$ if $C \subset C'$ and ν' is an extension of ν. It is not hard to see that every chain in C has an upper bound. By Zorn's Lemma there exists a maximal element (D, ν) in C.
Suppose $D \neq A$. Let $x \in A \setminus D$. Suppose D absorbs x. We define $\hat{\nu}$ on $D + \mathrm{co}\,\{x\}$ by

$$\hat{\nu}(d + \lambda x) = \begin{cases} \nu(d + \lambda x) & \text{if } d + \lambda x \in D, \\ \sup \nu & \text{if } d + \lambda x \notin D, \end{cases}$$

$(d \in D,\ \lambda \in B_K)$. It is not hard to verify that $\hat{\nu}$ is a norm on $D + \mathrm{co}\,\{x\}$. Property (iv) of Definition 3.2.1 is satisfied as D absorbs x. Now $\hat{\nu}$ is an extension of ν and hence an extension of $\| \ \|$. Furthermore, $\sup \hat{\nu} = \sup \nu = \sup \| \ \|$. Hence, $(D + \mathrm{co}\,\{x\}, \hat{\nu}) \in C$, $(D + \mathrm{co}\,\{x\} and \hat{\nu}) \succ (D, \nu)$. But it is not true that $(D + \mathrm{co}\,\{x\}, \hat{\nu}) = (D, \nu)$, a contradiction.
Thus, D does not absorb x. Then $\mathrm{co}\,\{x\}$ must be torsion free and the sum $D+ \mathrm{co}\,\{x\}$ is direct. We define $\hat{\nu} : D + \mathrm{co}\,\{x\} \to [0, \infty)$ by

$$\hat{\nu}(d + \lambda x) = \max(\nu(d), |\lambda| \sup \nu) \quad (d \in D, \lambda \in B_K).$$

It is not hard to see that $\hat{\nu}$ is a norm on $D + \mathrm{co}\,\{x\}$ and $(D + \mathrm{co}\,\{x\}, \hat{\nu}) \in C$, which leads again to a contradiction.
Hence, $D = A$ and ν is a norm on A, extending $\| \ \|$ and $\sup \nu = \sup \| \ \|$. □

3.2.6 Corollary *Every B_K-module can be provided with a (bounded) norm.*

Proof: Let A be a B_K-module. If $A = \{0\}$ then 0 is a norm on A. Suppose $A \neq \{0\}$. Let $x \in A$ with $x \neq \{0\}$. Let $\| \ \|$ on $\mathrm{co}\,\{x\}$ be defined by

$$\|\lambda x\| = \begin{cases} |\lambda| & \text{if } \lambda x \neq 0, \\ 0 & \text{if } \lambda x = 0, \end{cases}$$

($\lambda \in B_K$). It is easy to verify that $\| \ \|$ is a norm on $\operatorname{co}\{x\}$. By the previous theorem, $\| \ \|$ can be extended to a bounded norm ν on A. □

3.2.7 Definition Let A be a B_K-module and let $\| \ \|$ be a norm on A. A subset V of A is called $\| \ \|$-*open* if for every $y \in V$ there exists an $\varepsilon > 0$ such that $\{x \in A \mid \|x - y\| < \varepsilon\} \subset V$.
The $\| \ \|$-open subsets of A form the $\| \ \|$-*topology*.
Different from the usual terminology, we call a norm $\| \ \|$ *trivial* if the $\| \ \|$-topology is discrete.

3.2.8 Remark From (iv) of Definition 3.2.1 it follows that a norm on a B_K-module A can be trivial only if the valuation on K is trivial or A is a torsion module.
For example, let the valuation on K be non-trivial. Let $0 < r < 1$ and let $A = B_K/\mathrm{B}(0,r)$. (Recall that $\mathrm{B}(0,r) = \{\lambda \in B_K \mid |\lambda| < r\}$.) Let $\| \ \|$ on A be defined by

$$\|x\| = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \neq 0. \end{cases}$$

Then it is not hard to verify that property (i),(ii) and (iii) for a norm are satisfied. Property (iv) is also satisfied since if $x \in A$ and $\lambda_1, \lambda_2, \lambda_3, \ldots \in B_K$ such that $\lambda_n \to 0$, then $|\lambda_n| < r$ for large n, and hence $\|\lambda_n x\| = \|0\| = 0$ for large n. Thus, $\| \ \|$ is a trivial norm on A.

3.2.9 Proposition *Let A be a B_K-module and $\| \ \|$ a norm on A. Then $(A, \| \ \|)$ is a Hausdorff locally convex B_K-module.*

Proof: Let C be the collection $(\{x \in A \mid \|x\| < \varepsilon\})_{\varepsilon > 0}$. Then C is a collection of submodules of A. Trivially, for every finite subcollection $\mathcal{F}$ of C there exists a $U \in C$ such that $U \subset \bigcap \mathcal{F}$. If $|K|$ is non-trivial it follows from property (iv) of a norm that each $U \in C$ is absorbing. Furthermore, the $\| \ \|$-topology equals the C-topology. By Proposition 3.1.26 $(A, \| \ \|)$ is a locally convex B_K-module.
Let $y \in A$ with $y \neq 0$. Let $0 < \varepsilon < \|y\|$. Then $U := \{x \in A \mid \|x\| < \varepsilon\}$ is a zero neighbourhood and $y \notin U$. We see that $(A, \| \ \|)$ is Hausdorff. □

From vector space theory we know that every two norms on a finite dimensional K-vector space are equivalent. That is to say: all norms on a finite dimensional K-vector space induce the same topology. Furthermore, the vector space is complete with respect to (the unique) norm topology.
Surprisingly, the counterpart for B_K-modules is not true. There exist B_K-modules generated by one element admitting two non-equivalent norms. We will see that in the following example.

3.2.10 Example Let K be a field with a dense valuation. Let $0 < r < 1$. Let $\mathrm{B}(0,r) = \{\lambda \in B_K \mid |\lambda| \leq r\}$. Let $A = B_K/\mathrm{B}(0,r)$. Then $A = \operatorname{co}\{1 + \mathrm{B}(0,r)\}$ and hence A is 1-generated. Let $\| \ \|_1$ on A be defined by

$$\|x\|_1 = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \neq 0. \end{cases}$$

From the example in Remark 3.2.8 we know that $\| \; \|_1$ is a trivial norm on A. Let $\| \; \|_2$ on A be defined by

$$\|\lambda + \mathrm{B}(0,r)\|_2 = (|\lambda| - r) \vee 0 \quad (\lambda \in B_K).$$

We first show that $\| \; \|_2$ is a well defined map. To this end, let $\lambda, \mu \in B_K$ be such that $\lambda + \mathrm{B}(0,r) = \mu + \mathrm{B}(0,r)$.
If $|\lambda| \le r$, then $|\mu| \le r$ and hence $\|\lambda + \mathrm{B}(0,r)\|_2 = 0 = \|\mu + \mathrm{B}(0,r)\|_2$. If $|\lambda| > r$, then $|\lambda - \mu| \le r < \max(|\lambda|, |\mu|)$ and hence $|\lambda| = |\mu|$ which implies that $\|\lambda + \mathrm{B}(0,r)\|_2 = |\lambda| - r = |\mu| - r = \|\mu + \mathrm{B}(0,r)\|_2$.
It is not hard to verify that $\| \; \|_2$ is a norm on A. Now $\| \; \|_2$ is non-trivial, for let $\lambda_1, \lambda_2, \lambda_3 \in B_K$ be such that $|\lambda_1| > |\lambda_2| > |\lambda_3| > \dots$ and $\lim_{n\to\infty} |\lambda_n| = r$. Then $\lambda_n + \mathrm{B}(0,r) \to 0$ in the $\| \; \|_2$-topology and $\lambda_n + \mathrm{B}(0,r) \neq 0$ for every $n \in \mathbb{N}$.

3.2.11 Remark From K-vector space theory we know the following 'open mapping theorem'.

Let the valuation on K be non-trivial. Let $(E, \| \; \|)$ and $(F, \| \; \|')$ be Banach spaces. Let $T : (A, \| \; \|) \to (B, \| \; \|')$ be a continuous surjective homomorphism. Then T is open.

The counterpart of this theorem for B_K-modules is not true. In fact, there exist complete normed B_K-modules $(A, \| \; \|)$ and $(B, \| \; \|')$ and a continuous surjective homomorphism $T : A \to B$ that is not open.
For example, let K be spherically complete. Let $(B_K/\mathrm{B}(0,r), \| \; \|_2)$ be as in the previous example. (In Proposition 4.1.8 we will see that $(B_K/\mathrm{B}(0,r), \| \; \|_2)$ is complete.) It is easy to see that the canonical map $\pi : (B_K, | \; |) \to (B_K/\mathrm{B}(0,r), \| \; \|_2)$ is surjective and continuous. But, π is not open, since $\mathrm{B}(0,r)$ is open in $(B_K, | \; |)$ and $\pi(\mathrm{B}(0,r)) = \{0\}$, which is not open in $(B_K/\mathrm{B}(0,r), \| \; \|_2)$.

In Section 5.4 of Chapter 5 we will prove an open mapping-like theorem for complete normable locally compactoid B_K-modules.

Now we will work out an example of a B_K-module generated by one element with a non-trivial norm $\| \; \|$ on A for which $(A, \| \; \|)$ is not complete.

3.2.12 Example Let K be not spherically complete. Let $a_1, a_2, \dots \in B_K$ and $1 \ge r_1 > r_2 > \dots$ be such that $\mathrm{B}(a_1, r_1) \supset \mathrm{B}(a_2, r_2) \supset \dots$ and $\bigcap_{n\ge1} \mathrm{B}(a_n, r_n) = \emptyset$. Let $r = \lim_{n\to\infty} r_n$. Then $r > 0$, since K is complete. Let $A = B_K/\mathrm{B}(0,r)$. Then $A = \mathrm{co}\{1 + \mathrm{B}(0,r)\}$. Let the norm $\| \; \|$ on A be defined by $\|\lambda + \mathrm{B}(0,r)\| = (|\lambda| - r) \vee 0 \quad (\lambda \in B_K)$.
Now $a_1 + \mathrm{B}(0,r), a_2 + \mathrm{B}(0,r), \dots$ is a Cauchy sequence in A. For let $m, n \in \mathbb{N}$ with $m \ge n$. Then $a_m \in \mathrm{B}(a_n, r_n)$ which implies that $|a_m - a_n| < r_n$. Thus,

$$\|(a_m + \mathrm{B}(0,r)) - (a_n + \mathrm{B}(0,r))\| = \|(a_m - a_n) + \mathrm{B}(0,r)\| =$$
$$(|a_m - a_n| - r) \vee 0 \le r_n - r \to 0 \quad (n, m \to \infty).$$

Suppose that there exists a $y \in A$ such that $a_n + \mathrm{B}(0,r) \to y$. Let $x \in B_K$ be such that $y = x + \mathrm{B}(0,r)$. Then $(|a_n - x| - r) \vee 0 = \|(a_n - x) + \mathrm{B}(0,r)\| =$

$\|(a_n + \mathrm{B}(0,r)) - y\| \to 0 \quad (n \to \infty)$.
Let $m \in \mathbb{N}$. Then there exists an $N \in \mathbb{N}$ such that $|x - a_n| - r < r_m - r$ and hence $|x - a_n| < r_m$ for all $n > N$.
Let $n > \max(N, m)$. Then $|x - a_m| \leq \max(|x - a_n|, |a_n - a_m|) < r_m$. That is to say $x \in \mathrm{B}(a_m, r_m)$.
We obtain that $x \in \bigcap_{n\geq 1} \mathrm{B}(a_n, r_n) = \emptyset$, a contradiction.
Hence, $a_1 + \mathrm{B}(0,r)$, $a_2 + \mathrm{B}(0,r)$, $a_3 + \mathrm{B}(0,r), \ldots$ is a Cauchy sequence which does not converge. Thus $(A, \|\ \ \|)$ is not complete.

In the following section we will prove that every two norms on a torsion free B_K-module $A \in \mathcal{F}_K$ are equivalent and that A is complete with respect to the unique norm topology.

As promised in Remark 3.1.9 we now present an example of a complete normed torsion free B_K-module such that for every $\lambda \in B_K^-$ the map $x \mapsto \lambda x$ from A to λA is not a homeomorphism. Hence its norm is not equivalent to a faithful norm.

3.2.13 Example Let the valuation on K be non-trivial.
Let $A = B_K^{\mathbb{N}} = \{(\lambda_0, \lambda_1, \lambda_2, \ldots) \mid \lambda_i \in B_K \text{ for all } i \in \mathbb{N}\}$. Let $\|\ \ \|$ on A be defined by

$$\|(\lambda_0, \lambda_1, \lambda_2, \ldots)\| = |\lambda_0| \vee \sup_{n\geq 1} |\lambda_n|^n \quad ((\lambda_0, \lambda_1, \lambda_2, \ldots) \in B_K^{\mathbb{N}}).$$

It is not hard to verify that $\|\ \ \|$ is a norm on A.
We first show that $(A, \|\ \ \|)$ is complete. Let $(x^n)_{n\in\mathbb{N}}$ be a Cauchy sequence in A. Let $k \in \mathbb{N}$. Then $|x_k^n - x_k^m|^k \leq \|x^n - x^m\| \to 0$. Hence, also $x_k^n - x_k^m \to 0$ in B_K. As B_K is complete there exists a $y_k \in B_K$ such that $x_k^n \to y_k$. Set $y = (y_0, y_1, y_2, \ldots) \in B_K^{\mathbb{N}}$. We prove that $\lim_{n\to\infty} \|x^n - y\| = 0$.
Let $N \in \mathbb{N}$ such that $\|x^n - x^m\| < \varepsilon$ for all $m, n > N$. Let $k \in \mathbb{N}$. Let $m \geq N$ such that $|x_k^m - y_k|^k < \varepsilon$. Then for every $n \geq N$ we have that $|x_k^n - y_k|^k \leq (|x_k^n - x_k^m| \vee |x_k^m - y_k|)^k = |x_k^n - x_k^m|^k \vee |x_k^m - y_k|^k \leq \|x^n - x^m\| \vee |x_k^m - y_k|^k \leq \varepsilon \vee \varepsilon = \varepsilon$. Thus, $\|x^n - y\| = \sup_{k\in\mathbb{N}} |x_k^n - y_k|^k \leq \varepsilon$ for all $n \geq N$. Hence, $x^n \to y$ in $(A, \|\ \ \|)$.
We now show that for every $\lambda \in B_K^- \setminus \{0\}$ the map $x \mapsto \lambda x$ is not a homeomorphism from A to λA. Let $\lambda \in B_K^- \setminus \{0\}$. Let e_n be the element of A with a 1 on the n^{th} place and a 0 on the other places. Then $\|\lambda e_n\| = |\lambda|^n$ for each $n \geq 1$ and hence $\lambda e_n \to 0$ in the $\|\ \ \|$-topology. But not $e_n \to 0$ in the $\|\ \ \|$-topology since $\|e_n\| = 1$ for every $n \in \mathbb{N}$.
Let $U = \{(\lambda_0, \lambda_1, \lambda_2, \ldots) \in A \mid |\lambda_n| < 1 \text{ for large } n\}$. Then U is a submodule and $\{x \in A \mid \|x\| < 1\} \subset U$. Hence, U is open and therefore also closed. As $(A, \|\ \ \|)$ is complete it follows that also U is complete. Let $\lambda \in B_K^- \setminus \{0\}$. Let $x^n = (\underbrace{\lambda, \ldots, \lambda}_{n \text{ places}}, 0, 0, \ldots) \quad (n \in \mathbb{N})$ and let $x = (\lambda, \lambda, \lambda, \ldots)$.
Then $\|x - x^n\| = |\lambda|^n$ for all $n \geq 1$ and hence $\lim_{n\to\infty} \|x - x^n\| = 0$. As $x^n \in \lambda U$ for every $n \in \mathbb{N}$ it follows that $x \in \overline{\lambda U}$. But not $x \in \lambda U$. We obtain that λU is not closed. As λU is a submodule of A this implies that λU is also not open. Hence U is an example of an open and closed submodule of A for which λU is neither open nor closed for every $\lambda \in B_K^-$.

3.2.14 Remark We can extend the norm $\| \ \|$ on $B_K^{\mathbb{N}}$ to a norm $\| \ \|'$ on $K^{\mathbb{N}}$ by defining

$$\|(\lambda_0, \lambda_1, \lambda_2, \dots)\|' = (|\lambda_0| \wedge 1) \vee \sup_{n \ge 1}(|\lambda_n|^n \wedge 1) \quad ((\lambda_0, \lambda_1, \lambda_2, \dots) \in K^{\mathbb{N}}).$$

Then $(K^{\mathbb{N}}, \| \ \|')$ is a locally convex B_K-module but not a locally convex vector space in the usual sense. It is even not a topological vector space as we may see as follows. Let $e_0, e_1, e_2, \dots$ be the canonical unit vectors in $K^{\mathbb{N}}$. Let $\mu \in K$, $0 < |\mu| < 1$. Then $\mu e_n \to 0 \quad (n \to \infty)$. Now $\mu^{-1}\mu e_n = e_n$ for all $n \in \mathbb{N}$. Thus not $\mu^{-1}(\mu e_n) \to 0 \quad (n \to \infty)$.

3.2.15 Proposition *Let $(A, \| \ \|)$ be a normed B_K-module. Let B be a closed submodule of A. Then the map*

$$x + B \mapsto \inf_{b \in B} \|x - b\| \quad (x \in A)$$

is a norm on A/B; the so-called quotient norm.
If $\| \ \|$ is a Bosch norm then so is its quotient norm.
Moreover, the topology on A/B induced by the quotient norm is equal to the quotient topology of the $\| \ \|$-topology.

Proof: In this proof we will denote the quotient norm by $\| \ \|'$. We prove property (i)-(iv) of a norm for $\| \ \|'$.
(i) As B is closed we obtain for every $x \in A$ that

$$\|x + B\|' = 0 \iff \inf_{b \in B} \|x - b\| = 0 \iff x \in B.$$

Hence $\|y\|' = 0 \iff y = 0 \quad (y \in A/B)$.
(ii) Let $x_1, x_2 \in A$. For every $b_1, b_2 \in B$ we have that $b_1 + b_2 \in B$ and

$$\|(x_1 + x_2) - (b_1 + b_2)\| \le \max(\|x_1 - b_1\|, \|x_2 - b_2\|).$$

Therefore, $\|(x_1 + B) + (x_2 + B)\|' \le \max(\|x_1 + B\|', \|x_2 + B\|')$.
(iii) Let $x \in A$ and $\lambda \in B_K$. For every $b \in B$ we have that $\lambda b \in B$ and $\|\lambda x - \lambda b\| = \|\lambda(x - b)\| \le \|x - b\|$ and hence $\|\lambda(x + B)\|' \le \|x + B\|'$.
(iv) Let $x \in A$. Let $(\lambda_n)_{n \in \mathbb{N}} \in B_K$ be such that $\lambda_n \to 0 \quad (n \to \infty)$. Then $\|\lambda_n(x + B)\|' \le \|\lambda_n x - 0\| = \|\lambda_n x\| \to 0 \quad (n \to \infty)$.
If $\| \ \|$ on A is a Bosch norm then $\| \ \|'$ on A/B is also a Bosch norm, which can be seen as follows.
Let $x \in A$ and $\lambda \in B_K$. Then $\|\lambda x - \lambda b\| = \|\lambda(x - b)\| \le |\lambda| \|x - b\|$ for every $b \in B$. And hence $\|\lambda(x + B)\|' = \inf_{b \in B} \|\lambda x - b\| \le \inf_{b \in B} \|\lambda x - \lambda b\| \le \inf_{b \in B} |\lambda| \|x - b\| = |\lambda| \|x + B\|'$.
Finally, we prove that the $\| \ \|'$-topology and the quotient topology of the $\| \ \|$-topology coincide. To this end observe that for every $x \in A$ and every $\varepsilon > 0$

$$\|x + B\| < \varepsilon \iff x \in \{y \in A \mid \|y\| < \varepsilon\} + B.$$

This implies that $\pi(\{x \in A \mid \|x\| < \varepsilon\}) = \{w \in A/B \mid \|w\|' < \varepsilon\}$ for every $\varepsilon > 0$. From here we easily derive the assertion. □

3.2.16 Remark If A is a B_K-module, $\|\ \|$ a faithful norm on A and B a closed submodule of A then the quotient norm on A/B need not be faithful, even if A/B is torsion free. In the introduction of [17] one can find an example.

3.2.17 Proposition *Let $(A, \|\ \|)$ be a complete normed B_K-module. Let B be a closed submodule of A. Then the quotient topology on A/B is normable and complete.*

Proof: From Proposition 3.2.15 we obtain that the quotient topology on A/B is induced by the quotient norm $\|\ \|'$ on A/B. That $(A/B, \|\ \|')$ is complete can be proven in the same way as in the vectorial case. □

3.2.18 Remark In the next section we will prove that every norm on a B_K-module A is equivalent to a norm ν on A with $\sup \nu \le 1$. That is to say that for a norm $\|\ \|$ on a B_K-module A there exists a norm ν on A with $\sup \nu \le 1$ such that $\|\ \|$ and ν induce the same topology on A. From here we can prove the following.

Let $((A_n, \|\ \|_n))_{n\ge 1}$ be a collection of normed B_K-modules. Let $A = \prod_{n\in\mathbb{N}} A_n$. Then there exists a norm $\|\ \|$ on A such that the $\|\ \|$-topology equals the product topology on A.

A proof of this assertion can be given as follows. Let, for every $n \ge 1$, ν_n be a norm on A_n, with $\sup \nu_n \le 1$ such that the ν_n-topology and the $\|\ \|_n$-topology coincide. Let $\|\ \|$ on A be defined by

$$\|x\| = \max_{n\ge 1} \frac{1}{n} \nu_n(x(n)) \quad (x \in A).$$

Then $\|\ \|$ is a norm on A and from 2. of Proposition 3.1.15 it follows that the $\|\ \|$-topology equals the product topology on A.

3.3 Seminorms

3.3.1 Definition A *seminorm* on a B_K-module A is a map $p : A \to [0, \infty)$ with the following properties.

(i) $p(0) = 0$.

(ii) $p(x + y) \le \max(p(x), p(y)) \quad (x, y \in A)$.

(iii) $p(\lambda x) \le p(x) \quad (\lambda \in B_K,\ x \in A)$.

(iv) If $x \in A$ and $(\lambda_n)_{n\in\mathbb{N}} \in B_K$ such that $\lambda_n \to 0$ then $p(\lambda_n x) \to 0$.

A seminorm p is called a *Bosch seminorm* if p has property (i) and (ii) and in addition

(iii)′ $p(\lambda x) \le |\lambda| p(x) \quad (\lambda \in B_K,\ x \in A)$.

A seminorm p is called *faithful* if, besides (i) and (ii), p also satisfies (iii)″ $p(\lambda x) = |\lambda| p(x) \quad (\lambda \in B_K,\ x \in A)$.

3.3.2 Notation Let A be a B_K-module and p a seminorm on A. Let B be a submodule of A. Then the restriction of p, which is a seminorm on B, is denoted $p|B$.

To prevent confusion with seminorms on K-vector spaces we make the following definition.

3.3.3 Definition Let E be a K-vector space. A seminorm p on E is called a *vector space seminorm* if $p(\lambda x) = |\lambda| p(x)$ for every $\lambda \in K$ and every $x \in E$.

The following theorem can be proved in the same way as Theorem 3.2.5.

3.3.4 Theorem *Let A be a B_K-module and let B be a submodule of A. Let p be a bounded seminorm on B. Then there exists a seminorm q on A with* $\sup q = \sup p$ *such that $q|B = p$.*

3.3.5 Definition Let A be a B_K-module and let p be a seminorm on A. A subset V of A is called *p-open* if for every $y \in V$ there exists an $\varepsilon > 0$ such that $\{x \in A \mid p(x - y) < \varepsilon\} \subset V$. The collection of all p-open subsets is called the *p-topology*.

3.3.6 Proposition *Let A be a B_K-module and p a seminorm on A. Then the p-topology is a locally convex topology.*

The proof of this proposition is similar to that of Proposition 3.2.9. (Of course, we do not obtain the Hausdorff property). We will denote the B_K-module A provided with the p-topology by (A, p).

The following proposition is easy to verify.

3.3.7 Proposition *Let (A, τ) be a topological B_K-module and let p be a seminorm on A. Then:*
p is continuous $\iff$ p is continuous at 0 $\iff$ *For every $\varepsilon > 0$ the set $\{x \in A \mid p(x) < \varepsilon\}$ is open in A.*

3.3.8 Remark Openness of $\{x \in A \mid p(x) < 1\}$ does not guarantee continuity of p !

Equivalence of seminorms

The items 3.3.9 – 3.3.14 have their counterparts in vector space theory. The proofs are similar.

3.3.9 Definition Let A be a B_K-module and let p and q be seminorms on A. Then p is called *q-continuous* if p is continuous with respect to the q-topology.

3.3.10 Proposition *Let A be a B_K-module and let p and q be seminorms on A. Then the following assertions are equivalent.*

1. *p is q-continuous.*

2. *For every $\varepsilon > 0$ there exists a $\delta > 0$ such that*
$$\{x \in A \mid q(x) < \delta\} \subset \{x \in A \mid p(x) < \varepsilon\}.$$

3. *For every $\varepsilon > 0$ there exists a $\delta > 0$ such that*
$$\{x \in A \mid q(x) \le \delta\} \subset \{x \in A \mid p(x) \le \varepsilon\}.$$

4. *For every sequence $x_1, x_2, x_3, \ldots$ in A for which $x_n \to 0$ in the q-topology also $x_n \to 0$ in the p-topology (i.e. $q(x_n) \to 0$ implies $p(x_n) \to 0 \quad (n \to \infty)$).*

3.3.11 Proposition *Let A be a B_K-module and let p and q be seminorms on A such that p is q-continuous. Then* $\operatorname{Ker} q \subset \operatorname{Ker} p$.

3.3.12 Definition Let A be a B_K-module and let p and q be seminorms on A. Then p is called *equivalent* to q (notation: $p \sim q$) if the p-topology coincides with the q-topology.

3.3.13 Proposition *Let A be a B_K-module and let p and q be seminorms on A. Then $\quad p \sim q \iff p$ is q-continuous and q is p-continuous.*

3.3.14 Corollary *Let A be a B_K-module and let p and q be seminorms on A with $p \sim q$. Then* $\operatorname{Ker} p = \operatorname{Ker} q$.

In vector space theory two vector space seminorms p and q are equivalent iff there exist constants $c_1, c_2 > 0$ such that $c_1 p \le q \le c_2 p$. In our case (where seminorms need not be faithful) this statement no longer holds. Yet we shall derive a result that has the same spirit (Theorem 3.3.20 and Theorem 3.3.19). The first step is to compose increasing functions and seminorms.

3.3.15 Proposition *Let A be a B_K-module and let p be a seminorm on A. Let $f : \operatorname{conv}(p(A)) \to [0, \infty)$ be an increasing map such that $f(0) = 0$ and f is continuous at 0. Then $f \circ p$ is a seminorm on A and $f \circ p$ is p-continuous. If, in addition, $f(t) = 0 \iff t = 0$, then $f \circ p \sim p$.*

Proof: 1. $f(p)$ is a seminorm:

(i) $f(p(0)) = f(0) = 0$.

(ii) Let $x, y \in A$. As f is increasing we obtain that $f(p(x+y)) \le f(\max(p(x), p(y))) = \max(f(p(x)), f(p(y)))$.

(iii) Let $x \in A$ and $\lambda \in B_K$. As $p(\lambda x) \le p(x)$ and f is increasing we obtain that $f(p(\lambda x)) \le f(p(x))$.

(iv) Let $x \in A$ and $(\lambda_n)_{n\in\mathbb{N}} \in B_K$ such that $\lambda_n \to 0$. Then $p(\lambda_n x) \to 0$ and since f is continuous at 0 then also $f(p(\lambda_n x)) \to 0$.

2. $f \circ p$ is p-continuous:
Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A such that $x_n \to 0$ in the p-topology. Then $p(x_n) \to 0$ and as f is continuous at 0 we obtain that $f(p(x_n)) \to 0$ and hence $x_n \to 0$ in the $f \circ p$-topology.
Suppose that, in addition, $f(t) = 0 \iff t = 0$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A such that $x_n \to 0$ in the $f \circ p$-topology. Then $f(p(x_n)) \to 0$. Suppose not $p(x_n) \to 0$. Then there exists a $d > 0$ such that $p(x_n) > d$ for infinitely many n. Then $f(p(x_n)) \geq f(d)$ for infinitely many n and as $f(d) > 0$ this is in conflict to $f(p(x_n)) \to 0$. Hence, $p(x_n) \to 0$ and thus $x_n \to 0$ in the p-topology.
We obtain that p is $f \circ p$-continuous and hence $f \circ p \sim p$. □

3.3.16 Corollary *Let A be a B_K-module and let p be a seminorm on A. Then there exists a bounded seminorm q on A such that $p \sim q$.*

Proof: From the previous proposition we obtain that $\arctan \circ p$ is a bounded seminorm and $\arctan \circ p \sim p$. □

3.3.17 Lemma *Let A be a B_K-module and let p and q be seminorms on A. Then the following assertions are equivalent.*

(i) *p is q-continuous.*

(ii) *There exists an increasing function $\psi : \operatorname{conv}(p(A)) \to [0,\infty)$ with $\psi(t) = 0 \iff t = 0$ such that $\psi \circ p \leq q$.*

(ii)′ *There exists an increasing function $\tilde{\psi} : \operatorname{conv}(p(A)) \to [0,\infty)$ with $\tilde{\psi}(t) = 0 \iff t = 0$ and $\tilde{\psi}$ is continuous at 0 such that $\tilde{\psi} \circ p \leq q$.*

If, in addition, p is bounded then (i) *and* (ii) *are also equivalent with* (iii) *and* (iii)′ *below.*

(iii) *There exists an increasing function $\varphi : \operatorname{conv}(q(A)) \to [0,\infty)$ with $\varphi(0) = 0$ and φ is continuous at 0 such that $p \leq \varphi \circ q$.*

(iii)′ *There exists an increasing function $\tilde{\varphi} : \operatorname{conv}(q(A)) \to [0,\infty)$ with $\tilde{\varphi}(t) = 0 \iff t = 0$ and $\tilde{\varphi}$ is continuous at 0 such that $p \leq \tilde{\varphi} \circ q$.*

Proof: We first prove (i) ⇒ (ii) ⇒ (ii)′ ⇒ (i).
(i) ⇒ (ii): If $p = 0$ there is nothing to prove. So assume $p \neq 0$. Let $\delta \in \operatorname{conv}(p(A))$, $\delta > 0$. We define

$$U_\delta = \{\varepsilon > 0 \mid \{x \in A \mid q(x) < \varepsilon\} \subset \{x \in A \mid p(x) < \delta\}\}.$$

Now p is q-continuous, hence U_δ is not empty, even an interval. Since $\delta \in \operatorname{conv}(p(A))$ there exists a $y \in A$ with $p(y) \geq \delta$. Then $U_\delta \subset [0, q(y)]$, thus U_δ is bounded. Hence, we can define a map $\psi : \operatorname{conv}(p(A)) \to [0,\infty)$ by

$$\psi(\delta) = \begin{cases} 0 & \text{if } \delta = 0, \\ \sup U_\delta & \text{if } \delta > 0. \end{cases}$$

It is clear that ψ is increasing. We have already seen that for every $\delta \in \operatorname{conv}(q(A))$, $\delta > 0$ there exists an $\varepsilon > 0$ such that $\varepsilon \in U_\delta$. This implies that $\psi(t) = 0 \iff t = 0$. Moreover, $\psi \circ p \le q$. For let $y \in A$. If $q(y) = 0$ then, by Proposition 3.3.11, $p(y) = 0$ and hence $\psi(p(y)) = 0 \le q(y)$. Suppose $q(y) > 0$. For every $s > q(y)$ we obtain that $s \notin U_{p(y)}$ and hence $\psi(p(y)) \le q(y)$.
(ii) $\Rightarrow$ (ii)′: Let f be the identity map on $\operatorname{conv}(p(A))$. Then the function $\tilde{\psi} := \psi \wedge f : \operatorname{conv}(p(A)) \to [0,\infty)$ is increasing, since ψ and f are both increasing, $\tilde{\psi}(t) = 0 \iff \psi(t) = 0$ or $f(t) = 0 \iff t = 0$ and from $0 \le \tilde{\psi} \le f$ it follows that $\tilde{\psi}$ is continuous at 0. Furthermore, $\tilde{\psi} \le \psi$ and hence $\tilde{\psi} \circ p \le \psi \circ p \le q$. Hence, $\tilde{\psi}$ satisfies the requirements.
(ii)′ $\Rightarrow$ (i): Again, for $p = 0$ there is nothing to prove, so assume $p \ne 0$. Let $\varepsilon > 0$. We prove that there exists a $\delta > 0$ such that $\{x \in A \mid q(x) < \delta\} \subset \{x \in A \mid p(x) < \varepsilon\}$. If $\varepsilon \notin \operatorname{conv}(p(A))$ we are done. Thus we may assume $\varepsilon \in \operatorname{conv}(p(A))$. Then $\psi(\varepsilon) > 0$. Let $x \in A$ be such that $q(x) < \psi(\varepsilon)$. Then $\psi(p(x)) \le q(x) < \psi(\varepsilon)$ and since ψ is increasing we obtain that $p(x) < \varepsilon$. Hence we can take $\delta = \psi(\varepsilon)$.
From Proposition 3.3.10 we obtain that p is q-continuous.
Suppose p is bounded. We prove (i) $\Rightarrow$ (iii) $\Rightarrow$ (iii)′ $\Rightarrow$ (i).
(i) $\Rightarrow$ (iii): Let $\delta \in \operatorname{conv}(q(A))$. We define

$$V_\delta = \{\varepsilon \ge 0 \mid \{x \in A \mid q(x) \le \delta\} \subset \{x \in A \mid p(x) \le \varepsilon\}\}.$$

Let $c > 0$ such that $c > \sup p$. Then $c \in V_\delta$ and hence $V_\delta \ne \varnothing$ for every $\delta \in \operatorname{conv}(q(A))$. Thus we can define a map $\varphi : \operatorname{conv}(q(A)) \to [0,\infty)$ by $\varphi(\delta) = \inf V_\delta \quad (\delta \in \operatorname{conv}(q(A)))$. That φ is increasing is obvious. Furthermore, $\operatorname{Ker} q \subset \operatorname{Ker} p$ and hence $\varphi(0) = 0$. Moreover, φ is continuous at 0. For let $\varepsilon > 0$. As p is q-continuous there exists a $\delta \in \operatorname{conv}(q(A))$, with $\delta > 0$ such that $\{x \in A \mid q(x) \le \delta\} \subset \{x \in A \mid p(x) \le \varepsilon\}$. Then $\varphi(\delta) \le \varepsilon$. Since φ is increasing we obtain that $[0,\delta) \subset \varphi^{-1}([0,\varepsilon))$. Finally we prove $p \le \varphi \circ q$. In fact, let $y \in A$. For $s < p(y)$ we obtain that $y \in \{x \in A \mid q(x) \le q(y)\}$ but not $y \in \{x \in A \mid p(x) \le s\}$. Hence $s \notin V_{q(y)}$. Thus $V_{q(y)} \subset [p(y), c]$ and hence $\varphi(q(y)) \ge p(y)$.
(iii) $\Rightarrow$ (iii)′: Let f be the identity map on $\operatorname{conv}(q(A))$. Then the function $\tilde{\varphi} := \varphi \vee f : \operatorname{conv}(q(A)) \to [0,\infty)$ is increasing, $\tilde{\varphi}(t) = 0 \iff t = 0$ and $\tilde{\varphi}$ is continuous at 0. Furthermore, $\varphi \le \tilde{\varphi}$ and thus $p \le \varphi \circ q \le \tilde{\varphi} \circ q$. Hence, $\tilde{\varphi}$ satisfies the requirements.
(iii)′ $\Rightarrow$ (i): If $q = 0$ then $p = 0$ and we are done. Hence suppose $q \ne 0$. Let $\varepsilon > 0$. We have $\varphi(0) = 0$ and φ is continuous at 0, hence there exists a $\delta \in \operatorname{conv}(q(A))$ with $\delta > 0$ such that $\varphi(\delta) < \varepsilon$. Suppose $x \in A$ is such that $q(x) < \delta$. Then $p(x) \le \varphi(q(x)) < \varepsilon$. Thus,

$$\{x \in A \mid q(x) < \delta\} \subset \{x \in A \mid p(x) < \varepsilon\}.$$

By Proposition 3.3.10 p is q-continuous. □

3.3.18 Remark If the seminorm p in the previous proposition is not bounded there need not to exist a map φ as in (iii).
For example, let the valuation on K be non-trivial. Let $A = K^2$. Let

$p : A \to [0, \infty)$ be defined by

$$p((\lambda_1, \lambda_2)) = |\lambda_1| \vee |\lambda_2| \quad (\lambda_1, \lambda_2 \in K).$$

Let $q : A \to [0, \infty)$ be defined by

$$q((\lambda_1, \lambda_2)) = |\lambda_1| \vee \arctan |\lambda_2| \quad (\lambda_1, \lambda_2 \in K).$$

It is not hard to verify that p and q are equivalent (semi-)norms on A. Yet, (iii) does not hold. In fact, let $\lambda \in K$, $|\lambda| > 1$. Let $(x_n)_{n \in \mathbb{N}} \in A$ be defined by $x_n = (0, \lambda^n)$ $(n \in \mathbb{N})$. Then $p(x_n) = |\lambda|^n \to \infty$ $(n \to \infty)$. Should there exist an increasing map $\varphi : \operatorname{conv}(q(A)) \to [0, \infty)$ such that $p \le \varphi \circ q$ then $p(x_n) \le \varphi(q(x_n)) \le \varphi(\frac{1}{2}\pi)$ for all $n \in \mathbb{N}$, a contradiction.

The following two theorems are an immediate consequence of Lemma 3.3.17.

3.3.19 Theorem *Let A be a B_K-module and let p and q be seminorms on A. Then the following assertions are equivalent.*

(i) $p \sim q$.

(ii) *There exist increasing functions $\psi_1 : \operatorname{conv}(p(A)) \to [0, \infty)$ and $\psi_2 : \operatorname{conv}(q(A)) \to [0, \infty)$, both continuous at 0 and $\psi_1(t) = 0 \iff \psi_2(t) = 0 \iff t = 0$ such that $\psi_1 \circ p \le q$ and $\psi_2 \circ q \le p$.*

3.3.20 Theorem *Let A be a B_K-module and let p and q be* bounded *continuous seminorms on A. Then the following assertions are equivalent.*

(i) $p \sim q$.

(ii) *There exist increasing functions φ and $\psi : \operatorname{conv}(p(A)) \to [0, \infty)$, both continuous at 0 and $\varphi(t) = 0 \iff \psi(t) = 0 \iff t = 0$ such that $\psi \circ p \le q \le \varphi \circ p$.*

(iii) *There exist increasing functions φ and $\psi : \operatorname{conv}(q(A)) \to [0, \infty)$, both continuous at 0 and $\varphi(t) = 0 \iff \psi(t) = 0 \iff t = 0$ such that $\psi \circ q \le p \le \varphi \circ q$.*

In [30] one can find theorems like Theorem 3.3.19 and Theorem 3.3.20 for seminorms on topological groups.

Equivalence of norms on modules of finite rank

Warning: We remind the reader of the fact that norms need not be faithful so that, in particular on K-vector spaces, the concept of a norm is more general than the usual one.

Our purpose is to prove that all norms on a torsion free B_K-module of finite rank are equivalent. Before we start with this we first give a definition.

3.3.21 Definition Let $n \in \mathbb{N}$ and let E be an n-dimensional K-vector space. Let $e_1, \ldots, e_n$ be a (vector space) base for E. Then the vector space norm $\| \ \|_\infty$ on E, defined by

$$\|\lambda_1 e_1 + \cdots + \lambda_n e_n\|_\infty = \max_{1 \le i \le n} |\lambda_i| \quad (\lambda_1, \ldots, \lambda_n \in K),$$

is called the *max norm* with respect to $e_1, \ldots, e_n$.

3.3.22 Proposition *Let E be a one-dimensional K-vector space and let $\| \ \|$ be a norm on E. Let $e \in E$ such that $E = Ke$. Then there exists an increasing map $\varphi : [0, \infty) \to [0, \infty)$, with $\varphi(t) = 0 \iff t = 0$ and φ is continuous at 0, such that $\|\lambda e\| = \varphi(|\lambda|) \quad \lambda \in K$ Moreover, E is complete with respect to $\| \ \|$.*

Proof: Let $\lambda, \mu \in K$ such that $|\lambda| \ge |\mu|$. Then $\|\mu e\| = \|(\mu\lambda^{-1})\lambda e\| \le \|\lambda e\|$. Thus also, if $|\lambda| = |\mu|$ then $\|\lambda e\| = \|\mu e\|$. Hence we can define a map $\tilde{\varphi} : |K| \to [0, \infty)$ by $\tilde{\varphi}(|\lambda|) = \|\lambda e\| \quad (\lambda \in K)$. Then $\tilde{\varphi}$ is increasing and $\tilde{\varphi}(|\lambda|) = 0 \iff |\lambda| = 0$. We can extend $\tilde{\varphi}$ to a map $\varphi : [0, \infty) \to [0, \infty)$ by defining

$$\varphi(t) = \sup_{\substack{\pi \in |K| \\ \pi \le t}} \tilde{\varphi}(\pi) \quad (t \in [0, \infty)).$$

(If the valuation on K is trivial then there exists an $a > 0$ such that $\|E\| = \{0, a\}$. We then define $\varphi(t) = at \quad (t \in [0, \infty))$.) Then φ is is increasing, $\varphi(t) = 0 \iff t = 0$ and φ is continuous at 0. Furthermore, $\varphi(|\lambda|) = \|\lambda e\| \quad (\lambda \in K)$.
From Proposition 3.3.15 we obtain that $\| \ \|$ is equivalent with the norm $\lambda e \mapsto |\lambda|$ and E is complete with respect to this norm. □

3.3.23 Proposition *Let $n \in \mathbb{N}$. Let E be an n-dimensional K-vector space. Then all norms on E are equivalent and E is complete with respect to every norm.*

Proof: By induction. The case $n = 1$ is the previous proposition.
Let $n \in \mathbb{N}$ be such that all norms on a K-vector space E with $\dim E = n$ are equivalent and E is complete with respect to every norm. Let E be a K-vector space with $\dim E = n + 1$. Let $e_1, \ldots, e_{n+1}$ be a vector space base for E. Let $\| \ \|_\infty$ be the max norm with respect to $e_1, \ldots, e_{n+1}$.
Let $\| \ \|$ be a norm on E. We prove $\| \ \| \sim \| \ \|_\infty$. Let $(x_k)_{k \in \mathbb{N}}$ be a sequence in E. For each $k \in \mathbb{N}$ let $\lambda_1^k, \ldots, \lambda_{n+1}^k \in K$ such that $x_k = \lambda_1^k e_1 + \ldots + \lambda_{n+1}^k e_{n+1}$. Suppose $\|x_k\|_\infty \to 0$. Then $|\lambda_i^k| \to 0 \quad (k \to \infty)$ for every $i \in \{1, \ldots, n+1\}$ and hence $\|\lambda_i^k e_i\| \to 0 \quad (k \to \infty)$. Thus,

$$\|x_k\| = \|\lambda_1^k e_1 + \ldots + \lambda_{n+1}^k e_{n+1}\| \le \max_{1 \le i \le n+1} \|\lambda_i^k e_i\| \to 0 \quad (k \to \infty).$$

Suppose $\|x_k\| \to 0 \quad (k \to \infty)$. Suppose there exists an $i \in \{1, \ldots, n+1\}$ such that not $|\lambda_i^k| \to 0$. By symmetry we may assume that $i = n+1$. We may assume that there exists a $c > 0$ such that $|\lambda_{n+1}^k| > c \quad (k \in \mathbb{N})$.

Let $D = [e_1, \dots, e_n]$. By induction D is $\| \ \|$-complete and hence $\| \ \|$-closed. Let $\mu \in K$ such that $0 < |\mu| < c$. As $\mu e_{n+1} \notin D$ there exists an $\varepsilon > 0$ such that $\|x + \mu e_{n+1}\| > \varepsilon$ for all $x \in D$. In particular,

$$\|\frac{\mu}{\lambda_{n+1}^k}\lambda_1^k e_1 + \cdots + \frac{\mu}{\lambda_{n+1}^k}\lambda_n^k e_n + \mu e_{n+1}\| > \varepsilon \text{ for every } k \in \mathbb{N}.$$

Then $\|x_k\| \geq \|\frac{\mu}{\lambda_{n+1}^k} x_k\| = \|\frac{\mu}{\lambda_{n+1}^k}\lambda_1^k e_1 + \cdots + \frac{\mu}{\lambda_{n+1}^k}\lambda_n^k e_n + \mu e_{n+1}\| > \varepsilon$ for all $k \in \mathbb{N}$. This is in contradiction to $\|x_k\| \to 0$. Thus, $|\lambda_i^k| \to 0$ for every $i \in \{1, \dots, n+1\}$ and hence $\|x_k\|_\infty \to 0$.
From vector space theory we know that E is complete with respect to $\| \ \|_\infty$. As $\| \ \| \sim \| \ \|_\infty$ we obtain that also $(E, \| \ \|)$ is complete. □

3.3.24 Proposition *Let A be a B_K-module such that every two norms on A are equivalent. Let B be a submodule of A. Then every two norms on B are equivalent.*

Proof: Let $\| \ \|_1$ and $\| \ \|_2$ be norms on B. To show the equivalence we may, by Corollary 3.3.16, assume that $\| \ \|_1$ and $\| \ \|_2$ are bounded. Theorem 3.2.5 guarantees that there exist norms ν_1 and ν_2 on A such that $\nu_1|B = \| \ \|_1$ and $\nu_2|B = \| \ \|_2$. Then $\nu_1 \sim \nu_2$ and hence $\| \ \|_1 \sim \| \ \|_2$. □

3.3.25 Theorem *Let A be a torsion free B_K-module in $\mathcal{F}_K$. Then all norms on A are equivalent and A is complete with respect to every norm.*

Proof: Let $E = K \otimes_{B_K} A$ (see Remark 2.1.32). Then E is a finite dimensional vector space (see Remark 2.2.29). From Proposition 3.3.23 we obtain that all norms on E are equivalent. By using the previous proposition we obtain that every two norms on A are equivalent. To show completeness, let $e_1, \dots, e_n$ be a vector space base for E. Let $\| \ \|_\infty$ be the max norm with respect to $e_1, \dots, e_n$ on E. By Proposition 2.1.31, A is an absorbing submodule of E. Thus A is open and hence closed with respect to $\| \ \|_\infty$. Since E is complete with respect to $\| \ \|_\infty$ we obtain that A is complete with respect to the restriction of $\| \ \|_\infty$ to A. As all norms on A are equivalent we obtain that A is complete with respect to any norm. □

We have seen in Example 3.2.10 that even on 1-generated B_K-modules we may have inequivalent norms. In the following proposition we will see that there exist at most two equivalent classes of norms on a rank 1 B_K-module.

3.3.26 Proposition *Let $|K|$ be non-trivial. Let A, B be absolutely convex subsets of K such that $\{0\} \subsetneqq B \subsetneqq A$. Let $V = \operatorname{conv}|B|$. Let $\| \ \|$ be a norm on A/B.*

1. *If $|K|$ is discrete or $|K|$ is dense and $V = [0, c)$ with $c \in |K|$ then $\| \ \|$ is trivial.*

2. *If $|K|$ is dense and $V = [0, c]$ with $c \in |K|$ or $V = [0, c)$ with $c \notin |K|$ then $\| \ \|$ is trivial or $\| \ \|$ is equivalent with $\| \ \|'$, where $\| \ \|'$ is defined by $\|\lambda + B\|' = (|\lambda| - c) \vee 0 \quad (\lambda \in A)$.*

Proof: 1. Let $\lambda \in A$ such that $\lambda \notin B$ and $\mu \in B$ for every $\mu \in A$ with $|\mu| < |\lambda|$. Let $d = \|\lambda + B\|$. Then $d > 0$. Let $x \in A/B$, $x \neq 0$. There exist a $\mu \in A$ such that $x = \mu + B$. Then $|\mu| \geq |\lambda|$ and hence $d = \|\lambda + B\| = \|(\lambda\mu^{-1})(\mu + B)\| \leq \|\mu + B\| = \|x\|$. Then $\{x \in A/B \mid \|x\| < d\} = \{0\}$ and hence $\{0\}$ is open which is to say that $\|\ \|$ is trivial.
2. Suppose $\|\ \|$ is non-trivial. Let $\varepsilon > 0$. Let $U = \{x \in A/B \mid \|x\|' < \varepsilon\}$. Then U is a non-zero submodule of A/B. Let $y \in U$, $y \neq 0$. Let $\lambda \in A$ such that $y = \lambda + B$. Let $\delta = \|y\| > 0$. Let $x \in A/B$ such that $\|x\| < \delta$. Let $\mu \in A$ such that $x = \mu + B$. Then $|\mu| < |\lambda|$. (If $|\mu| \geq |\lambda|$ then $\lambda\mu^{-1} \in B_K$ and hence $\|y\| = \|\lambda + B\| = \|(\lambda\mu^{-1})(\mu + B)\| \leq \|\mu + B\| = \|x\| < \delta$, a contradiction.) Then $x = (\mu\lambda^{-1})y \in U$.
We obtain that $\{x \in A/B \mid \|x\| < \delta\} \subset \{x \in A/B \mid \|x\|' < \varepsilon\}$. By using Proposition 3.3.10 we obtain that $\|\ \|'$ is $\|\ \|$-continuous. Observe that we used only the fact that $\|\ \|'$ is non-trivial. So by symmetry we obtain that $\|\ \|$ is $\|\ \|'$-continuous. Hence, $\|\ \| \sim \|\ \|'$. □

3.3.27 Corollary *Let A be a torsion B_K-module of rank 1. Then all non-trivial norms on A are equivalent.*

3.3.28 Remark The previous corollary is in general not true for B_K-modules of rank n with $n > 1$. For example, Let $K = \mathbb{C}_p$. Let $A = B_K/pB_K \times B_K/pB_K$. Then rank $A = 2$.
Let ν_1 on B_K/pB_K be defined by $\nu_1(\lambda + pB_K) = (|\lambda| - p) \vee 0 \quad (\lambda \in B_K)$ and let ν_2 be a trivial norm on B_K/pB_K. Let $\|\ \|_1$ on A be defined by

$$\|(x_1, x_2)\|_1 = \max(\nu_1(x_1), \nu_2(x_2)) \quad (x_1, x_2 \in B_K/pB_K)$$

Then $\|\ \|_1$ and $\|\ \|_2$ are both non-trivial norms but they are not equivalent.

3.3.29 Remark It is an interesting open problem whether for a B_K-module A that is a product of two B_K-modules or rank 1 there exist at most four equivalence classes of norms.

Some Special Seminorms

3.3.30 Proposition *Let (A, τ) be a locally convex B_K-module and let U be a submodule of A. If $|K|$ is non-trivial, U is supposed to be absorbing. Then $p_U : A \to [0, \infty)$ defined by*

$$p_U(x) = \begin{cases} 0 & \text{if } x \in U, \\ 1 & \text{if } x \notin U \end{cases}$$

is a seminorm and: p_U is continuous $\iff$ U is open.

Proof: 1. As U is a submodule it is easy to verify that p_U satisfies the properties (i),(ii) and (iii) for a seminorm. If $|K|$ is trivial, property (iv) is automatically satisfied. If $|K|$ is non-trivial, property (iv) is satisfied since U is absorbing. Hence, p_U is a seminorm.
2. We prove p_U is continuous $\iff$ U is open.

$\Rightarrow$) Suppose p_U is continuous. Then $U = \{x \in A \mid p_U(x) < 1\} = p_U^{-1}([0,1))$. And the latter set is open since p_U is continuous. Hence, U is open.
$\Leftarrow$) Suppose U is open. Then $U \subset \{x \in A \mid p_U(x) < \varepsilon\}$ for every $\varepsilon > 0$ and hence the latter set is open for all $\varepsilon > 0$. By Proposition 3.3.7, p_U is continuous. □

3.3.31 Proposition *Let A be a B_K-module and let U be an absorbing submodule of A. Then the so-called* Minkowsky function $q_U : A \to [0,\infty)$ *defined by*

$$q_U(x) = \inf\{|\lambda| \mid \lambda \in K, x \in \lambda U\}$$

is a Bosch seminorm on A.

Proof:

(i) $q_U(0) = 0$, since $0 \in 0U$.

(ii) Let $x, y \in A$. To show that $q_U(x+y) \leq \max(q_U(x), q_U(y))$ we may assume that $q_U(x) \geq q_U(y)$. Set $t = q_U(x)$.
If $|K|$ is discrete then there exists a $\lambda \in K$ such that $|\lambda| = t$. Then $x \in \lambda U$ and $y \in \lambda U$ and hence also $x + y \in \lambda U$. Thus $q_U(x+y) \leq |\lambda| = t = \max(q_U(x), q_U(y))$.
If $|K|$ is dense then $x, y \in \lambda U$ for every $\lambda \in K$ with $|\lambda| > t$. Then also $x + y \in \lambda U$ for every $\lambda \in K$ with $|\lambda| > t$. Hence $q_U(x+y) \leq |\lambda|$ for every $\lambda \in K$ with $|\lambda| > t$. As $|K|$ is dense we obtain that $q_U(x+y) \leq t = \max(q_U(x), q_U(y))$.

(iii)′ Let $x \in A$ and $\mu \in B_K$. To prove $q_U(\mu x) \leq |\mu| q_U(x)$, set $q_U(x) = t$.
If $|K|$ is discrete then there exists a $\lambda \in K$ with $|\lambda| = t$. Then $x \in \lambda U$ and hence $\mu x \in \mu(\lambda U) \subset (\mu\lambda)U$. Thus $q_U(\mu x) \leq |\mu||\lambda| = |\mu| q_U(x)$.
If $|K|$ is dense then $x \in \lambda U$ for every $\lambda \in K$ with $|\lambda| > t$. Then $\mu x \in \mu(\lambda U) \subset (\mu\lambda)U$ for all $\lambda \in K$, $|\lambda| > t$. Hence $q_U(\mu x) \leq |\mu||\lambda|$ for every $\lambda \in K$ with $|\lambda| > t$. Since $|K|$ is dense we obtain that $q_U(\mu x) \leq |\mu| t = |\mu| q_U(x)$.

□

3.3.32 Remark Observe that q_U depends on the embedding module A.

3.3.33 Proposition *Let A be a B_K-module and let U be an absorbing submodule of A. Then*

1. $p_U \leq q_U$.

2. *For every* $\lambda \in B_K$: $\{x \in A \mid q_U(x) < |\lambda|\} \subset \lambda U \subset \{x \in A \mid q_U(x) \leq |\lambda|\}$.

3. *If τ is a locally convex topology on A then* q_U *continuous* $\Rightarrow$ U *is* τ*-open.*

The proof is straightforward.

3.3.34 Remark It is not surprising that the converse of 3. above is not true as we see from the following example.
Let $K = \mathbb{C}_p$, let $\mathrm{B}(0, \frac{1}{p}) = \{\lambda \in K \mid |\lambda| \leq \frac{1}{p}\}$. Let $A = B_K/\mathrm{B}(0, \frac{1}{p})$. Let the norm $\|\ \|$ on A be defined as

$$\|\lambda + \mathrm{B}(0, \frac{1}{p})\| = (|\lambda| - \frac{1}{p}) \vee 0 \quad (\lambda \in B_K).$$

Let τ be the $\|\ \|$-topology.
Now $\{x \in A \mid q_A(x) < \frac{1}{p}\} = \{0\}$, for if $q_A(x) < \frac{1}{p}$, then $x \in pA = \{0\}$.
We see that $\{x \in A \mid q_A(x) < \frac{1}{p}\}$ is not open and hence q_A is not continuous (while A is obviously open).

3.3.35 Proposition *Let (A, τ) be a locally convex B_K-module and let U be a submodule of A. If $|K|$ is non-trivial then U is supposed to be absorbing. Let $\overline{p}_U : A \to [0, \infty)$ be defined by*

$$\overline{p}_U(x) = \sup_{\substack{\lambda \in B_K \\ \lambda \neq 0}} |\lambda|^{-1} p_U(\lambda x) \quad (x \in A).$$

Then

1. $\overline{p}_U(x) < \infty$ *for all* $x \in A$.
2. $\overline{p}_U(x) = 0 \iff p_U(x) = 0 \iff x \in U$.
3. $\overline{p}_U(x)$ *is a Bosch seminorm.*
4. *the* $\overline{p}_U$*-topology equals the* p_U*-topology.*
5. $\overline{p}_U$ *is the smallest among all Bosch seminorms* r *for which* $r \geq p_U$.

Proof: If $|K|$ is trivial then $\overline{p}_U = p_U$ and there is nothing to prove, so suppose $|K|$ is non-trivial.
1. Let $x \in A$. As U is absorbing there exists an $r \in (0, 1)$ such that $\lambda x \in U$ for every $\lambda \in B_K$ with $|\lambda| < r$. And hence $p_U(\lambda x) = 0$ for every $\lambda \in B_K$ with $|\lambda| < r$. Then

$$\overline{p}_U(x) = \sup_{\substack{\lambda \in B_K \\ |\lambda| \geq r}} |\lambda|^{-1} p_U(\lambda x) \leq \sup_{\substack{\lambda \in B_K \\ |\lambda| \geq r}} |\lambda|^{-1} \leq r^{-1} < \infty.$$

2. Let $x \in A$. Then $\overline{p}_U(x) = 0 \iff p_U(\lambda x) = 0$ for every $\lambda \in B_K \iff p_U(x) = p_U(1 \cdot x) = 0 \iff x \in U$.
3. (i) $\overline{p}_U(0) = 0$, since $p_U(\lambda 0) = 0$ for every $\lambda \in B_K$.
(ii) Let $x, y \in A$. We show $\overline{p}_U(x + y) \leq \max(p_U(x), p_U(y))$. By symmetry we may assume that $\overline{p}_U(x) \geq \overline{p}_U(y)$. Set $\overline{p}_U(x) = t$. If $t = 0$ then $x, y \in U$ and hence $x+y \in U$ which means that $\overline{p}_U(x+y) = 0 = \max(p_U(x), p_U(y))$. Suppose $t > 0$. Then $p_U(\lambda x) = 0$ and $p_U(\lambda y) = 0$ for every $\lambda \in B_K$ with

$|\lambda| < t^{-1}$ and hence also $p_U(\lambda(x+y)) = 0$ for every $\lambda \in B_K$ with $|\lambda| < t^{-1}$. Then

$$\overline{p}_U(x+y) \leq \sup_{\substack{\lambda \in B_K \\ |\lambda| \geq t^{-1}}} |\lambda|^{-1} p_U(\lambda(x+y)) \leq t = \max(\overline{p}_U(x), \overline{p}_U(y)).$$

(iii)′ Let $x \in A$ and $\mu \in B_K$. We prove $\overline{p}_U(\mu x) \leq |\mu| \overline{p}_U(x)$. Set $\overline{p}_U(x) = t$. If $t = 0$ we are done.
Suppose $t > 0$. Then $p_U(\lambda x) = 0$ for every $\lambda \in B_K$ with $|\lambda| < t^{-1}$. If $|\mu| < t^{-1}$ then $p_U(\mu x) = 0$ and hence $\overline{p}_U(\mu x) = 0 \leq |\mu| \overline{p}_U(x)$. If $|\mu| \geq t^{-1}$ then $p_U(\lambda(\mu x)) = 0$ for every $\lambda \in B_K$ with $|\lambda| < |\mu|^{-1} t^{-1}$ and hence

$$\overline{p}_U(\mu x) \leq \sup_{\substack{\lambda \in B_K \\ |\lambda| \geq |\mu|^{-1} t^{-1}}} |\lambda|^{-1} p_U(\lambda(\mu x)) \leq |\mu| t = |\mu| \overline{p}_U(x).$$

4. $\{x \in A \mid p_U(x) < \varepsilon\} = \{x \in A \mid \overline{p}_U(x) < \varepsilon\} = U$ for every $\varepsilon \in (0,1)$. It follows that the $\overline{p}_U$-topology equals the p_U-topology.
5. Let $x \in A$. Then $\overline{p}_U(x) \geq |1|^{-1} p_U(1.x) = p_U(x)$. Hence, $\overline{p}_U \geq p_U$.
Suppose q is a Bosch seminorm on A with $q \geq p_U$. Let $x \in A$. For every $\lambda \in B_K \setminus \{0\}$ we obtain that $q(x) \geq |\lambda|^{-1} q(\lambda x) \geq |\lambda|^{-1} p_U(\lambda x)$ and hence

$$q(x) \geq \sup_{\substack{\lambda \in B_K \\ \lambda \neq 0}} |\lambda|^{-1} p_U(\lambda x) = \overline{p}_U(x).$$

Thus, $q \geq \overline{p}_U$. □

Now we connect $\overline{p}_U$ with the Minkowsky function q_U of Definition 3.3.31. If the valuation on K is trivial then the only absorbing submodule of a B_K-module A is A itself and q_A is a trivial norm on A. If the valuation on K is non-trivial then, for every absorbing submodule U of A, q_U is a Bosch seminorm with $q_U \geq p_U$. Hence $q_U \geq \overline{p}_U$. But there is a much more direct relation between q_U and $\overline{p}_U$, as we see in the next proposition.

3.3.36 Proposition *Let $|K|$ be non-trivial. Let A be a B_K-module and let U be an absorbing submodule of A.*
If the valuation on K is dense then

$$\overline{p}_U(x) = \begin{cases} 0 & \text{if } x \in U, \\ q_U(x) & \text{if } x \notin U. \end{cases}$$

If the valuation on K is discrete and then

$$\overline{p}_U(x) = \begin{cases} 0 & \text{if } x \in U, \\ \pi q_U(x) & \text{if } x \notin U, \end{cases}$$

where $\pi = \max\{|\lambda| \mid \lambda \in B_K, |\lambda| < 1\}$.

Proof: 1. Suppose that $|K|$ is dense. Let $x \in A$. If $x \in U$ then $\overline{p}_U(x) = 0$ as we have already seen before. If $x \notin U$ then $\overline{p}_U(x) \geq 1$. Let $s > \overline{p}_U(x)$. Then

there exists a $\mu \in K$ such that $s > |\mu| > \overline{p}_U(x)$. This implies $p_U(\mu^{-1}x) = 0$, in other words $\mu^{-1}x \in U$. Then $|\mu| \in \{|\lambda| \mid \lambda \in K,\ x \in \lambda U\}$ and hence $q_U(x) \leq |\mu| < s$.
We obtain that $q_U(x) \leq \overline{p}_U(x)$. And thus, by 5. of Proposition 3.3.35, $q_U(x) = \overline{p}_U(x)$.
2. Suppose $|K|$ is discrete. Let $x \in A$. If $x \in U$ then of course $\overline{p}_U(x) = 0$. If $x \notin U$ then $\overline{p}_U(x) \geq 1$. Let $\mu \in K$ such that $\overline{p}_U(x) = |\mu|$. Then $p_U(\mu^{-1}x) = 1$ and hence $x \notin \mu U$. That is to say $|\mu| \notin \{|\lambda| \mid \lambda \in K,\ x \in \lambda U\}$. Let $\nu \in B_K$ such that $|\nu| = \pi$. Then $\overline{p}_U(x) < |\nu|^{-1}|\mu|$ and hence, $p_U(\nu\mu^{-1}x) = 0$. This implies $x \in (\nu^{-1}\mu)U$ and hence, $|\nu|^{-1}|\mu| \in \{|\lambda| \mid \lambda \in K,\ x \in \lambda U\}$. Then $q_U(x) = |\nu|^{-1}|\mu|$ and hence $\overline{p}_U(x) = |\nu| q_U(x) = \pi q_U(x)$. □

3.4 Seminorms and Locally Convex Topologies

3.4.1 Definition Let A be a B_K-module and let $\mathcal{P}$ be a collection of seminorms A. A subset V of A is called *$\mathcal{P}$-open* if for every $y \in V$ there exist $n \in \mathbb{N}$, $p_1, \ldots, p_n \in \mathcal{P}$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that

$$\{x \in A \mid p_i(x - y) < \varepsilon_i \quad (i = 1, \ldots, n)\} \subset V.$$

The collection of all $\mathcal{P}$-open subsets of A is called the *$\mathcal{P}$-topology*.

3.4.2 Proposition *Let A be a B_K-module and let $\mathcal{P}$ be a non-empty collection of seminorms on A. Then the $\mathcal{P}$-topology is a locally convex topology on A.*

Proof: The collection

$$C = (\{x \in A \mid p_i(x) < \varepsilon_i \quad (i = 1, \ldots, n)\})_{n \in \mathbb{N},\ p_1, \ldots, p_n \in \mathcal{P},\ \varepsilon_1, \ldots, \varepsilon_n > 0}$$

is a collection of submodules of A such that for every finite subcollection $\mathcal{F}$ of C there exists a $U \in C$ such that $U \subset \bigcap \mathcal{F}$. If $|K|$ is non-trivial then, by property (iv) for a seminorm, every member of C is absorbing. Furthermore, the C-topology equals the $\mathcal{P}$-topology. Now apply Proposition 3.1.26. □

3.4.3 Proposition *Let (A, τ) be a locally convex B_K-module and let $\mathcal{P}$ be the collection of all continuous seminorms on A. Then τ equals the $\mathcal{P}$-topology.*

Proof: That every $\mathcal{P}$-open submodule is τ-open is an easy consequence of Proposition 3.3.7. Let U be a τ-open submodule of A. Then the seminorm p_U is τ-continuous and hence $U = \{x \in A \mid p_U(x) < 1\}$ is $\mathcal{P}$-open. □

3.4.4 Definition Let $\mathcal{P}$ be a collection of seminorms on a locally convex B_K-module (A, τ). We say that $\mathcal{P}$ *generates* τ if the $\mathcal{P}$-topology equals τ.

The following proposition has its counterpart in vector space theory. The proof is similar.

3.4.5 Proposition *Let (A, τ) be a locally convex B_K-module.*

(i) *Let $\mathcal{P}$ be a generating collection of seminorms for τ. Let $(x_\alpha)_{\alpha\in I}$ be a net in A. Then*

$$x_\alpha \to 0 \iff p(x_\alpha) \to 0 \text{ for every } p \in \mathcal{P}.$$

(ii) *If $\mathcal{P}$ is a collection seminorms on A such that for each net $(x_\alpha)_{\alpha\in I}$ in A*

$$x_\alpha \to 0 \iff p(x_\alpha) \to 0 \text{ for every } p \in \mathcal{P},$$

then $\mathcal{P}$ is a generating collection of seminorms for τ.

3.4.6 Definition Let A be a B_K-module. A collection $\mathcal{P}$ of seminorms on A is called *separating* if for every $x \in A$ with $x \neq 0$ there exists a $p \in \mathcal{P}$ such that $p(x) \neq 0$.

3.4.7 Proposition *Let (A, τ) be a locally convex topology. Then (A, τ) is Hausdorff $\iff$ There exists a separating collection $\mathcal{P}$ of seminorms on A such that $\tau = \mathcal{P}$-topology.*

An analogous theorem is known from vector space theory. The proof is similar.

3.4.8 Proposition *Let (A, τ) be a locally convex B_K-module. Let*

$$\mathcal{P} = \{p_U\}_{U \text{ open submodule of } A}.$$

Then the $\mathcal{P}$-topology equals τ.

Proof: Every $q \in \mathcal{P}$ is continuous and hence the $\mathcal{P}$-topology is weaker than τ. In the proof of Proposition 3.4.3 we have already seen that every τ-open submodule is $\mathcal{P}$-open. □

3.4.9 Corollary *Every locally convex topology on a B_K-module A is generated by a collection of bounded seminorms.*

3.4.10 Proposition *Every locally convex topology on a B_K-module A is generated by a collection of Bosch seminorms.*

Proof: For every open submodule U of A the seminorm $\overline{p}_U$ is a Bosch seminorm. Let $\mathcal{P} = \{\overline{p}_U \mid U \text{ open submodule of } A\}$. From Proposition 3.3.35 we know that $p_U \sim \overline{p}_U$ for every open submodule U of A. By using Proposition 3.4.8 we obtain that τ equals the $\mathcal{P}$-topology. □

3.4.11 Proposition *Let (A, τ) be a locally convex B_K-module and let B be a submodule of A. If $\mathcal{P}$ is a collection of seminorms on A generating τ, then $\mathcal{P}' := \{p|B \mid p \in \mathcal{P}\}$ is a collection of seminorms on B generating $\tau|B$.*

Proof: For every $p \in \mathcal{P}$ and every $\varepsilon > 0$ we have $\{x \in B \mid p|B(x) < \varepsilon\} = \{x \in A \mid p(x) < \varepsilon\} \cap B$. From here we easily derive the assertion. □

3.4.12 Proposition *Let I be an index set and for every $i \in I$ let (A_i, τ_i) be a locally convex B_K-module. Let $A = \prod_{i \in I} A_i$ and let τ be the product topology on A. For each $i \in I$ let $\mathcal{Q}_i$ be a collection seminorms on A_i generating τ_i. Then $\mathcal{Q} := (p \circ P_i)_{p \in \mathcal{Q}_i,\, i \in I}$ is a collection seminorms on A generating τ.*

Proof: Combine Proposition 3.1.15 and Proposition 3.4.5. □

3.4.13 Remark Let (A, τ) be a locally convex B_K-module and let B be a submodule of A. For a seminorm p on A we define the seminorm p' on A/B by

$$p'(x + B) = \inf_{b \in B} p(x - b) \quad (x \in A).$$

Suppose $\mathcal{P}$ is a collection of seminorms on A generating τ. Then the collection $\mathcal{P}' := \{p' \mid p \in \mathcal{P}\}$ need not to generate the quotient topology on A/B. (Compare Proposition 3.2.15).
For example, let $K = \mathbb{Q}_p$ and let $A = B_K \times B_K$. Let the seminorms p_1 and p_2 on A be defined by

$$p_1((\lambda_1, \lambda_2)) = |\lambda_1 - \lambda_2| \quad (\lambda_1, \lambda_2 \in B_K),$$
$$p_2((\lambda_1, \lambda_2)) = |\lambda_1 + \lambda_2| \quad (\lambda_1, \lambda_2 \in B_K).$$

Let $\mathcal{P} = \{p_1, p_2\}$ and let τ be the $\mathcal{P}$-topology on A. Let $B = \{0\} \times B_K \subset A$. Then

$$p_1'((\lambda_1, \lambda_2) + B) \le p_1((\lambda_1, \lambda_2) - (0, \lambda_2 - \lambda_1)) = p_1(\lambda_1, \lambda_1) = 0$$

for every $\lambda_1, \lambda_2 \in B_K$. Hence, $p_1' = 0$ on A/B. In the same way we obtain that $p_2' = 0$ on A/B. Hence, $\mathcal{P}' = \{0\}$. Now

$$U := \{x \in A \mid p_1(x) < \frac{1}{p} \text{ and } p_2(x) < \frac{1}{p}\}$$

is τ-open and hence $\pi(U)$ is open in the quotient topology on A/B. (Here $\pi : A \to A/B$ is the quotient map.) Now $\pi(U) \neq A/B$, since, for example, $(1, 0) + B \notin \pi(U)$. This implies that $\pi(U)$ is not $\mathcal{P}'$-open. Thus $\mathcal{P}'$ does not generate the quotient topology.

Later on in this section we will introduce the notion of a weak base of seminorms for a locally convex topology. There we call a collection $\mathcal{P}$ of seminorms on a locally convex B_K-module (A, τ) a weak base of seminorms for τ if the collection $\{x \in A \mid p(x) < \varepsilon\}_{p \in \mathcal{P},\, \varepsilon > 0}$ is a base of seminorms for τ. Then we have the following.

Let (A, τ) be a locally convex B_K-module and let B be a submodule of A. Let $\mathcal{P}$ be a weak base of seminorms for τ. Then $\mathcal{P}' := \{p' \mid p \in \mathcal{P}\}$ is a weak base of seminorms for the quotient topology of τ on A/B.

Proof: Let $q \in \mathcal{P}'$ and let $\varepsilon > 0$. We prove that $\{x \in A/B \mid q(x) < \varepsilon\}$ is open in the quotient topology. To this end let $p \in \mathcal{P}$ such that $q = p'$. Then

$$\{x \in A/B \mid q(x) < \varepsilon\} = \pi(\{y \in A \mid p(y) < \varepsilon\}$$

and the latter set is open in the quotient topology.
Now we prove that the collection $\{x \in A/B \mid q(x) < \varepsilon\}_{q \in \mathcal{P}',\, \varepsilon>0}$ is a base of zero neighbourhoods for the quotient topology. To this end let U be a submodule of A/B that is open in the quotient topology. Then there exists a τ-open submodule V of A such that $\pi(V) = U$. As $\mathcal{P}$ is a weak base of seminorms for τ we obtain that there exists a $p \in \mathcal{P}$ and an $\varepsilon > 0$ such that $\{y \in A \mid p(y) < \varepsilon\} \subset V$. Then $p' \in \mathcal{P}'$ and

$$\{x \in A/B \mid p'(x) < \varepsilon\} = \pi(\{y \in A \mid p(x) < \varepsilon\}) \subset \pi(V) = U.$$

□

3.4.14 Proposition *Let (A, τ) be a locally convex B_K-module. Then: (A, τ) is metrizable $\iff$ There exist a countable, separating collection $\mathcal{P}$ of seminorms on A such that $\tau = \mathcal{P}$-topology.*

An analogous proposition is known in vector space theory. The proof is similar. But since our notion of norm is more 'relaxed' we can even prove the following.

3.4.15 Theorem *Let (A, τ) be a locally convex B_K-module. Then: (A, τ) is metrizable $\iff$ There exists a norm $\|\ \|$ on A such that τ equals the $\|\ \|$-topology.*

Proof: $\Rightarrow$) Suppose (A, τ) is metrizable. From the previous lemma we obtain that there exist a separating collection $\mathcal{P} = \{p_1, p_2, p_3, \dots\}$ of seminorms on A such that $\tau = \mathcal{P}$-topology. We may suppose that each p_n is bounded and $\sup p_n \le 1$ for all $n \ge 1$. Otherwise we take $p_n \wedge 1$ instead of p_n for all $n \ge 1$ ($p_n \wedge 1 = \varphi \circ p_n$, where $\varphi : \operatorname{conv}(p_n(A)) \to [0, \infty)$ is defined by $\varphi(t) = t \wedge 1 \quad (t \in \operatorname{conv}(p_n(A)))$ and hence, by Proposition 3.3.15, $p_n \wedge 1$ is a seminorm and $p_n \wedge 1 \sim p$ for all $n \in \mathbb{N}$). Let

$$q = p_1 \vee \frac{1}{2}p_2 \vee \frac{1}{3}p_3 \vee \dots .$$

One verifies easily that q is a seminorm on A and, as $\mathcal{P}$ is separating, even a norm. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in A. Then

$$q(x_n) \to 0 \iff p_k(x_n) \to 0 \text{ for all } k \in \mathbb{N}.$$

By using Proposition 3.4.5 we obtain that the q-topology equals τ.
$\Leftarrow$) is obvious. □

3.4.16 Proposition *Let (A, τ) be a Hausdorff locally convex B_K-module. Let $\mathcal{P}$ be a collection of seminorms on A that generates τ. Let for every $p \in \mathcal{P}$ the norm $\overline{p}$ on $A/\operatorname{Ker} p$ be defined by $\overline{p}(x + \operatorname{Ker} p) = p(x) \quad (x \in A)$. Then (A, τ) is topologically embeddable in $\prod_{p \in \mathcal{P}}(A/\operatorname{Ker} p, \overline{p})$, provided with the product topology.*

The proof of this proposition is similar to that of Proposition 3.1.31 and is omitted.

3.4.17 Corollary *Every Hausdorff locally convex B_K-module is embeddable in a product of normed B_K-modules.*

Continuous Seminorms

First we prove a general proposition on continuous seminorms.

3.4.18 Proposition *Let $\mathcal{P}$ be a collection of seminorms on a B_K-module A. Then:*

1. *Let $p \in \mathcal{P}$ and let $\varphi : \operatorname{conv}(p(A)) \to [0, \infty)$ be an increasing function such that $\varphi(t) = 0 \iff t = 0$ and φ is continuous at 0. Then $\varphi \circ p$ is a $\mathcal{P}$-continuous seminorm.*
2. *Let $p, q \in \mathcal{P}$. Then $p \vee q$ is a $\mathcal{P}$-continuous seminorm.*
3. *Let $p_1, p_2, p_3, \ldots \in \mathcal{P}$ and let p be such that $\lim_{n\to\infty} p_n = p$ uniformly on A. Then p is a $\mathcal{P}$-continuous seminorm.*

Proof: **1.** From Proposition 3.3.15 we obtain that $\varphi \circ p \sim p$. Hence $\varphi \circ p$ is $\mathcal{P}$-continuous.
2. It is not hard to see that $p \vee q$ is a seminorm. That $p \vee q$ is $\mathcal{P}$-continuous is even more obvious.
3. First we show that p is a seminorm.

(i) $p(0) = \lim_{n\to\infty} p_n(0) = \lim_{n\to\infty} 0 = 0$.

(ii) Let $x, y \in A$. Let $\varepsilon > 0$. Let $n \in \mathbb{N}$ be such that $\sup_{x\in A} |p_n(x) - p(x)| < \frac{1}{2}\varepsilon$. Then $p(x+y) \le p_n(x+y) + \frac{1}{2}\varepsilon \le \max(p_n(x), p_n(y)) + \frac{1}{2}\varepsilon \le \max(p(x) + \frac{1}{2}\varepsilon, p(y) + \frac{1}{2}\varepsilon) + \frac{1}{2}\varepsilon = \max(p(x), p(y)) + \varepsilon$.
We obtain $p(x+y) \le \max(p(x), p(y))$.

(iii) Let $x \in A$ and $\lambda \in B_K$. Let $\varepsilon > 0$. Let $n \in \mathbb{N}$ be such that $\sup_{x\in A} |p_n(x) - p(x)| < \frac{1}{2}\varepsilon$.
Then $p(\lambda x) \le p_n(\lambda x) + \frac{1}{2}\varepsilon \le p_n(x) + \frac{1}{2}\varepsilon \le p(x) + \varepsilon$.
We obtain $p(\lambda x) \le p(x)$.

(iv) Let $x \in A$ and let $(\lambda_n)_{n\in\mathbb{N}} \in B_K$ be such that $\lambda_n \to 0$. Let $\varepsilon > 0$. Let $m \in \mathbb{N}$ be such that $\sup_{x\in A} |p_m(x) - p(x)| < \frac{1}{2}\varepsilon$. Then $p_m(\lambda_n x) \le \frac{1}{2}\varepsilon$ for large n and hence $p(\lambda_n x) < \varepsilon$ for large n.
We obtain $p(\lambda_n x) \to 0 \quad (n \to \infty)$.

The proof of the $\mathcal{P}$-continuity of p is standard. □

Now we prove a lemma on extension of seminorms whose counterpart in vector space theory is well-known.

3.4.19 Lemma *Let A be a B_K-module and let B be a submodule of A. Let p be a seminorm on B and let q be a seminorm on A such that $p \le q$ on B. Then $r : A \to [0, \infty)$, defined by*

$$r(x) = \inf_{b\in B} \max(p(b), q(x-b)) \quad (x \in A),$$

is a seminorm on A, $r \le q$ and $r = p$ on B.

Proof: **1.** Let $x \in A$. Then $r(x) \le \max(p(0), q(x-0)) = q(x)$, hence $r \le q$.
2. It is easy to see that $r \ge 0$. We prove that r is a seminorm. We only prove property (iii) and (iv) for a seminorm.

(iii) Let $x \in A$ and let $\lambda \in B_K$. Let $\varepsilon > 0$. Let $b \in B$ be such that $\max(p(b), q(x-b)) < r(x) + \varepsilon$. Then also $\lambda x \in B$ and $r(\lambda x) \le \max(p(\lambda b), q(\lambda x - \lambda b)) \le \max(p(b), q(x-b)) < r(x) + \varepsilon$.
We obtain that $r(\lambda x) \le r(x)$.

(iv) Let $x \in A$ and let $\lambda_1, \lambda_2, \lambda_3, \ldots \in B_K$ be such that $\lim_{n\to\infty} \lambda_n = 0$. Now $r(\lambda_n x) \le q(\lambda_n x)$ and $q(\lambda_n x) \to 0 \quad (n \to \infty)$. Hence, also $r(\lambda_n x) \to 0 \quad (n \to \infty)$.

3. Let $x \in B$. Then $r(x) \le \max(p(x), q(x-x)) = p(x)$. On the other hand, $p(x) \le \max(p(b), p(x-b)) \le \max(p(b), q(x-b))$ for all $b \in B$ and hence $p(x) \le \inf_{b\in B}(p(b), q(x-b)) = r(x)$. Thus, $r = p$ on B. □

3.4.20 Theorem *Let (A, τ) be a locally convex B_K-module and let B be a submodule of A. Let p be a bounded seminorm on B that is continuous with respect to $\tau|B$. Then there exists a bounded τ-continuous seminorm q on A with $q|B = p$ and* $\sup q = \sup p$.

Proof: We first prove that there exists a continuous seminorm r on A such that $p \le r$ on B. Let $s = \sup p$. Let $V_n = \{x \in B \mid p(x) < \frac{1}{n}\} \quad (n \ge 1)$. It is an open submodule of B for every $n \ge 1$. By Proposition 3.1.30 for every $n \ge 1$ there exists an open submodule U_n of A such that $V_n = U_n \cap B$. As $V_1 \supset V_2 \supset V_3 \supset \ldots$ we may assume that $U_1 \supset U_2 \supset U_3 \supset \ldots$. Let

$$r_n = s p_{U_1} \vee p_{U_2} \vee \frac{1}{2} p_{U_3} \vee \ldots \vee \frac{1}{n-1} p_{U_n} \quad (n \ge 1)$$

and

$$r = s p_{U_1} \vee p_{U_2} \vee \frac{1}{2} p_{U_3} \vee \ldots \ .$$

Then each r_n is continuous and $r = \lim_{n\to\infty} r_n$ uniformly on A. By 3. of Proposition 3.4.18, r is a τ-continuous seminorm on A. Now $p \le r$ on B. For let $x \in B$. If $p(x) = 0$ we are done. Suppose $p(x) > 0$. If $x \notin V_1$ then $x \notin U_1$ and hence $r(x) = s \ge p(x)$. If $x \in V_1$ then let $n \in \mathbb{N}$ be such that $x \in V_n$ and not $x \in V_{n+1}$. Then $x \in U_n \setminus U_{n+1}$ and hence $r(x) = \frac{1}{n} > p(x)$, since $x \in V_n$.
Now we define $q : A \to [0, \infty)$ by

$$q(x) = \inf_{b\in B} \max(p(b), r(x-b)) \quad (x \in A).$$

Then q is a seminorm on A, $q \le r$ and $q = p$ on B. And q is continuous since $q \le r$. Furthermore, $\sup q = s = \sup p$. □

3.4.21 Remark For an unbounded continuous seminorm on a submodule there may be no extension to the entire module. For example, let the valuation on K be dense. Let $(A, \tau) = (B_K, |\ |)$. Let ν on B_K^- be defined by

$$\nu(\lambda) = \frac{|\lambda|}{1 - |\lambda|} \quad (\lambda \in B_K^-).$$

By Proposition 3.3.15, ν is a (semi)norm on B_K^- and $\nu \sim |\ |$. Hence, ν is $\tau|B_K^-$-continuous. But, obviously there does not exist a (semi)norm p on B_K such that $p|B_K^- = \nu$.

3.4.22 Theorem *Let (A, τ) be a locally convex B_K-module, let B be a closed submodule of A and let $x \in A \setminus B$. Then there exists a continuous seminorm p on A with $p \leq 1$ such that $p(B) = \{0\}$ and $p(x) = 1$.*

Proof: Let $q : B + \operatorname{co}\{x\} \to [0, \infty)$ be defined by

$$q(y + \mu x) = \begin{cases} 1 & \text{if } |\mu| = 1, \\ 0 & \text{if } |\mu| < 1, \end{cases}$$

$(y \in B,\ \mu \in B_K)$. Then q is well-defined. To this end suppose $y, z \in B$ and $\lambda, \mu \in B_K$ such that $y + \lambda x = z + \mu x$. Then $(\lambda - \mu)x = z - y \in B$. Hence, $|\lambda - \mu| < 1$, since $x \notin B$. Thus $|\lambda| = 1 \iff |\mu| = 1$ and hence $q(y + \lambda x) = q(z + \mu x)$. It is not hard to verify that q is a seminorm on $B + \operatorname{co}\{x\}$. Furthermore, q is continuous with respect to $\tau|B + \operatorname{co}\{x\}$. For let $(z_\alpha)_{\alpha \in I}$ be a net in $B + \operatorname{co}\{x\}$ such that $z_\alpha \xrightarrow{\tau} 0$. Let $(y_\alpha)_{\alpha \in I}$ be a net in B and $(\mu_\alpha)_{\alpha \in I}$ a net in B_K such that $z_\alpha = y_\alpha + \mu_\alpha x \quad (\alpha \in I)$. Suppose not $|\mu_\alpha| < 1$ for large α. Then there exists a cofinal subset J of I such that $z_\alpha = y_\alpha + \mu_\alpha x$ and $|\mu_\alpha| = 1$ for all $\alpha \in J$. Then $(\mu_\alpha^{-1})_{\alpha \in J}$ is a net in B_K and $(\mu_\alpha^{-1} z_\alpha)_{\alpha \in J}$ is a net in $B + \operatorname{co}\{x\}$. From Proposition 3.1.29 we obtain that $\mu_\alpha^{-1} z_\alpha \xrightarrow{\tau} 0$. That is to say that $\mu_\alpha^{-1} y_\alpha + x \xrightarrow{\tau} 0$. This implies that $\mu_\alpha^{-1} y_\alpha \to -x$, and hence $x \in \overline{B}$, a contradiction. Thus, $|\mu_\alpha| < 1$ for large α and hence $q(z_\alpha) = 0$ for large α. By the previous theorem there exists a continuous seminorm p on A which is an extension of q. Then $p|B = q|B = 0$ and $p(x) = q(x) = 1$. □

3.4.23 Proposition *Let (A, τ) be a locally convex B_K-module and let B be a B_K-module such that $A \subset B$. Then there exists a locally convex topology σ on B such that $\sigma|A = \tau$. If τ is Hausdorff we may choose σ to be Hausdorff too.*

Proof: Let $\mathcal{P}$ be the collection

$$\{p \mid p \text{ is a bounded seminorm on } B \text{ and } p|A \text{ is } \tau\text{-continuous}\}.$$

Let σ be the $\mathcal{P}$-topology. Then σ is a locally convex topology.
Let $\mathcal{P}' = \{p|A \mid p \in \mathcal{P}\}$. From Proposition 3.4.11 we obtain that $\sigma|A$ equals the $\mathcal{P}'$-topology. By using Theorem 3.3.4 we obtain that $\mathcal{P}'$ equals the collection of all bounded continuous seminorms on A. Thus, $\sigma|A = \tau$.
Suppose τ is Hausdorff. We prove that σ is Hausdorff. Let $x \in B$. If $x \in A$ then there exists a bounded τ-continuous seminorm p on A with $p(x) > 0$.

There exists a σ-continuous seminorm r with $r|A = p$. Then $r(x) > 0$. If $x \notin A$ then we define $p : A + \operatorname{co}\{x\} \to [0, \infty)$ as follows.

$$p(y + \lambda x) = \begin{cases} 1 & \text{if } |\lambda| = 1, \\ 0 & \text{if } |\lambda| < 1, \end{cases}$$

($y \in A$, $\lambda \in B_K$). Then p is a bounded seminorm on $A + \operatorname{co}\{x\}$. Hence there exists a bounded seminorm r on B such that $r|(A + \operatorname{co}\{x\}) = p$. Then $r(x) = 1 > 0$. Now $r|A = p|A = 0$ and hence $r|A$ is τ-continuous. This implies $r \in \mathcal{P}$.
Wee see that $\mathcal{P}$ is separating. Now use Proposition 3.4.7. □

A Base of Seminorms for a Locally Convex Topology

We first introduce the notions weakly saturated and saturated.

3.4.24 Definition Let A be a B_K-module. A collection $\mathcal{P}$ of bounded seminorms on A is called *weakly saturated* if $\mathcal{P}$ has the following three properties.

1. For all $n \in \mathbb{N}$: if $p_1, \dots, p_n \in \mathcal{P}$ then also $p_1 \vee \dots \vee p_n \in \mathcal{P}$.
2. If $p \in \mathcal{P}$ and $c > 0$ then also $cp \in \mathcal{P}$.
3. If $p \in \mathcal{P}$ and q is a seminorm on A such that $q \le p$, then also $q \in \mathcal{P}$.

A collection $\mathcal{P}$ of bounded seminorms on A is called *saturated* if $\mathcal{P}$ is weakly saturated and

4. If $p_1, p_2, p_3, \dots \in \mathcal{P}$ and $p = \lim_{n\to\infty} p_n$ uniformly on A then also $p \in \mathcal{P}$.

In locally convex vector space theory the requirements 1., 2. and 3. in the previous definition already imply that $\mathcal{P}$ is the collection of all continuous seminorms. Here we need an extra requirement, like 4., as we see in the following proposition. (In fact, requirement 4. is always satisfied in vector space theory, see also the subsection *Base of Seminorms for Locally Convex Vector Spaces*, page 90.)

3.4.25 Proposition *Let A be a B_K-module and let $\mathcal{P}$ be a collection of bounded seminorms on A. Let τ be the $\mathcal{P}$-topology. Then*

1. *$\mathcal{P}$ is weakly saturated $\Rightarrow$ $p_U \in \mathcal{P}$ for every τ-open submodule U of A.*
   (For the definition of p_U see Proposition 3.3.30.)
2. *$\mathcal{P}$ is saturated $\Rightarrow$ $\mathcal{P}$ is the collection of all bounded τ-continuous seminorms.*

Proof: 1. Suppose $\mathcal{P}$ is weakly saturated. Let U be an open submodule of A. There exist an $n \in \mathbb{N}$, $p_1, \dots, p_n \in \mathcal{P}$ and $\varepsilon_1, \dots, \varepsilon_n > 0$ such that $\{x \in A \mid p_i(x) < \varepsilon_i \quad (i = 1, \dots, n)\} \subset U$. As $\mathcal{P}$ is weakly saturated we obtain that $q = \frac{1}{\varepsilon_1} p_1 \vee \dots \vee \frac{1}{\varepsilon_n} p_n \in \mathcal{P}$. Then $\{x \in A \mid q(x) < 1\} \subset U$ and

hence $p_U \leq q$. By using again that $\mathcal{P}$ is weakly saturated we obtain $p_U \in \mathcal{P}$.
2. Suppose $\mathcal{P}$ is saturated. Let p be a bounded τ-continuous seminorm on A. We prove that $p \in \mathcal{P}$. In fact, let $U_n = \{x \in A \mid p(x) < \frac{1}{n+1}\}$ $(n \in \mathbb{N})$. Then U_n is open and hence, by part one of the proof, $p_{U_n} \in \mathcal{P}$ for every $n \in \mathbb{N}$. Let $s = \sup p$ and let $q_n = sp_{U_0} \vee 1p_{U_1} \vee \ldots \vee \frac{1}{n}p_{U_n}$ $(n \in \mathbb{N})$. Then $q_n \in \mathcal{P}$ for every $n \in \mathbb{N}$ since $\mathcal{P}$ is saturated. Let $q = sp_{U_0} \vee 1p_{U_1} \vee \frac{1}{2}p_{U_2} \vee \ldots$. Then $q = \lim_{n\to\infty} q_n$ uniformly on A and hence $q \in \mathcal{P}$.
Now $p \leq q$. For let $x \in A$. If $p(x) = 0$ we are done. Suppose $p(x) > 0$. If $x \notin U_0$ then $q(x) \geq sp_{U_0}(x) = s \geq p(x)$.
If $x \in U_0$ then let $n \geq 1$ such that $x \in U_{n-1} \setminus U_n$. Then $p(x) < \frac{1}{n} = q(x)$.
As $\mathcal{P}$ is saturated we obtain that $p \in \mathcal{P}$. □

The following is an example of a collection seminorms on B_K that is weakly saturated but not saturated.

3.4.26 Example Let the valuation on K be non-trivial. Let $\mathcal{P}$ be the collection of all bounded seminorms q on B_K such that there exist a $c > 0$ and an $\varepsilon > 0$ such that $q \leq c \cdot p_{B(0,\varepsilon)}$. It takes a standard verification to see that $\mathcal{P}$ is weakly saturated. Since $\operatorname{Ker} p \neq \{0\}$ for every $q \in \mathcal{P}$, we obtain that $\mathcal{P}$ is not saturated. In fact, $|\ |$ is a bounded $\mathcal{P}$-continuous seminorm on B_K, but not a member of $\mathcal{P}$.

Now we shall introduce bases of seminorms for locally convex B_K-modules and compare them to the corresponding notion in locally convex K-vector spaces (see page 90).

3.4.27 Definition Let (A, τ) be a locally convex B_K-module. A collection $\mathcal{P}$ of seminorms on A is called a *strong base of seminorms* for τ if

1. $\tau = \mathcal{P}$-topology,

2. for every bounded continuous seminorm q on A there exists a $p \in \mathcal{P}$ and a $c > 0$ such that $q \leq cp$.

In Definition 3.4.35 we shall define weak bases of seminorms. In the following example we will see that a strong base of seminorms need not be weakly saturated.

3.4.28 Example Let the valuation on K be non-trivial. We consider the B_K-module $(B_K, |\ |)$. Let Ψ be the set of all increasing maps $\varphi : [0,1] \to [0,\infty)$ such that $\varphi(t) = 0 \iff t = 0$ and φ is continuous at 0. Let the collection $\mathcal{P}$ of seminorms on B_K be given by

$$\mathcal{P} = \{\varphi \circ |\ | \mid \varphi \in \Psi\}.$$

By Proposition 3.3.15 each $q \in \mathcal{P}$ is $|\ |$-continuous and hence the $\mathcal{P}$-topology equals the $|\ |$-topology. Now let p be a bounded $|\ |$-continuous seminorm on B_K. By Lemma 3.3.17, there exists a map $\varphi \in \Psi$ such that $p \leq \varphi \circ |\ |$. We see that $\mathcal{P}$ is a strong base of seminorms for the $|\ |$-topology.

As $\operatorname{Ker} q = \{0\}$ for every $q \in \mathcal{P}$ we have that $p_U \in \mathcal{P}$ for no $|\ |$-open submodule U of A and hence, by Proposition 3.4.25, the collection $\mathcal{P}$ is not weakly saturated.

3.4.29 Theorem *Let (A, τ) be a locally convex B_K-module and let $\mathcal{P}$ be a collection of seminorms on A such that*

1. *$\tau = \mathcal{P}$-topology,*

2. *$\mathcal{P}$ is closed under finite suprema and uniform limits,*

3. *if $p \in \mathcal{P}$ and $\varphi : \operatorname{conv}(p(A)) \to [0, \infty)$ is an increasing function such that $\varphi(t) = 0 \iff t = 0$ and φ is continuous at 0. Then also $\varphi \circ p \in \mathcal{P}$.*

Then $\mathcal{P}$ is a strong base of seminorms for τ.

Proof: First we observe the following. Let $p \in \mathcal{P}$ and let $a > 0$. Let $\varphi : \operatorname{conv}(p(A)) \to [0, \infty)$ be defined by $\varphi(t) = at \quad (t \in [0, \infty))$. Then $\varphi \circ p = ap$ and hence, according to 3., $ap \in \mathcal{P}$.
Let q be a bounded continuous seminorm on A. We prove that there exists a $p \in \mathcal{P}$ such that $q \leq p$. For each $k \in \mathbb{N}$ let $V_k = \{x \in A \mid q(x) < \frac{1}{k+1}\}$. Now each V_k is a $\mathcal{P}$-open, hence there exist $n \in \mathbb{N}$, $p_1, \ldots, p_n \in \mathcal{P}$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that $\{x \in A \mid p_1(x) < \varepsilon_1, \ldots, p_n(x) < \varepsilon_n\} \subset V_k$. Then $\frac{1}{\varepsilon_i} p_i \in \mathcal{P}$ for every $i \in \{1, \ldots, n\}$ and hence $\frac{1}{\varepsilon_1} p_1 \vee \ldots \vee \frac{1}{\varepsilon_n} p_n \in \mathcal{P}$ and $\{x \in A \mid (\frac{1}{\varepsilon_1} p_1 \vee \ldots \vee \frac{1}{\varepsilon_n} p_n)(x) < 1\} \subset V_k$.
We see: for every $k \in \mathbb{N}$ there exists a $q_k \in \mathcal{P}$ with $\{x \in A \mid q_k(x) < 1\} \subset V_k$. We may assume that $\sup q_k \leq 1$ for all $k \in \mathbb{N}$ otherwise we take $\varphi(q_k)$ instead of q_k, where $\varphi : \operatorname{conv}(q_k(A)) \to [0, \infty)$ is defined by

$$\varphi(t) = \begin{cases} t & \text{if } t < 1, \\ 1 & \text{if } t \geq 1. \end{cases}$$

Set $s = \sup q$. Let

$$p_k = sq_0 \vee 1q_1 \vee \ldots \vee \frac{1}{k} q_k \quad (k \in \mathbb{N})$$

and

$$p = sq_0 \vee 1q_1 \vee \frac{1}{2} q_2 \vee \ldots \ .$$

Then $p = \lim_{k \to \infty} p_k$ uniformly on A and hence $p \in \mathcal{P}$. In the same way as in 2. of the proof of Proposition 3.4.25 one verifies that $q \leq p$. □

In the proof we have seen that we can, without harm, substitute property 3. by the following two properties.

3a. If $p \in \mathcal{P}$ and $c > 0$ then $cp \in \mathcal{P}$.

3b. If $p \in \mathcal{P}$ then also $p \wedge 1 \in \mathcal{P}$.

3.4.30 Remark This conditions on $\mathcal{P}$ for being a strong base of seminorms on A are minimal conditions in the following sense.

1) If (A, τ) is a locally convex B_K-module and $\mathcal{P}$ is a collection of bounded seminorms such that only

1. τ equals the $\mathcal{P}$-topology,
2. $\mathcal{P}$ is closed under finite suprema and uniform limits,
3. $cp \in \mathcal{P}$ for every $p \in \mathcal{P}$ and every $c > 0$.

Then $\mathcal{P}$ is not necessarily a strong base of seminorms for τ. For example, let the valuation on K be non-trivial. Let $A = B_K$ and let $\mathcal{P} = \{c \cdot | \ | \mid c > 0\}$. Let τ be the $| \ |$-topology.
Then $\mathcal{P}$ satisfies property 1. , 2. and 3. but $\sqrt{| \ |}$ is a bounded $\mathcal{P}$-continuous seminorm and there does not exist a $p \in \mathcal{P}$ and a $c > 0$ such that $\sqrt{| \ |} \le cp$.

2) Let (A, τ) be a locally convex B_K-module and let $\mathcal{P}$ be a collection of bounded seminorms on A such that only

1. $\tau = \mathcal{P}$-topology,
2. $\mathcal{P}$ is closed under finite suprema,
3. if $p \in \mathcal{P}$ and $\varphi : \operatorname{conv}(p(A)) \to [0, \infty)$ is an increasing function such that $\varphi(t) = 0 \iff t = 0$ and φ is continuous at 0. Then also $\varphi \circ p \in \mathcal{P}$.

Then $\mathcal{P}$ is not necessarily a strong base of seminorms for τ. For example, let (A, τ) be a metrizable locally convex B_K-module such that there exists a base of zero neighbourhoods, $(U_n)_{n \ge 1}$, consisting of submodules of A such that such that $U_1 \supsetneq U_2 \supsetneq U_2 \supsetneq \dots$.
(For instance, $(A, \tau) = (K, | \ |)$, where the valuation on K is non-trivial.)
Let $\mathcal{P} = \{a_1 p_{U_1} \vee \dots \vee a_n p_{U_n} \mid n \in \mathbb{N}, a_1, \dots, a_n > 0\}$. Then
1. τ equals the $\mathcal{P}$-topology:
For every $n \in \mathbb{N}$ we have that $\{x \in A \mid (p_{U_1} \vee \dots \vee p_{U_n})(x) < 1\} = U_n$ and hence U_n is open in the $\mathcal{P}$-topology. We obtain that τ is weaker than the $\mathcal{P}$-topology.
Let $q \in \mathcal{P}$ and let $\varepsilon > 0$. There exist an $n \in \mathbb{N}$ and $a_1, \dots, a_n > 0$ such that $q = a_1 p_{U_1} \vee \dots \vee a_n p_{U_n}$. Then $q(x) = 0$ for every $x \in U_n$ and hence $U_n \subset \{x \in A \mid q(x) < \varepsilon\}$. We obtain that the $\mathcal{P}$-topology is weaker than τ.
2. $\mathcal{P}$ is closed with respect to finite suprema:
Let $p_1, \dots, p_n \in \mathcal{P}$.
There exist $k_1, \dots, k_n \in \mathbb{N}$ and $a_1^1, \dots a_{k_1}^1, \dots, a_1^n, \dots, a_{k_n}^n > 0$ such that

$$p_i = a_1^i p_{U_1} \vee \dots \vee a_{k_i}^i p_{U_{k_i}} \quad (k = 1, \dots, n).$$

Let $m = \max(k_1, \dots, k_n)$.
Define $a_j^i = 0$ for $j \in \{k_i + 1, \dots, m\}$, $(i = 1, \dots, n)$.
Let $c_j = \max(a_j^1, \dots, a_j^n) \quad (j = 1, \dots, m)$. Then

$$p_1 \vee \dots \vee p_n = c_1 p_{U_1} \vee \dots \vee c_m p_{U_m} \in \mathcal{P}.$$

3. For every $q \in \mathcal{P}$ and every increasing function $\varphi : \operatorname{conv}(q(A)) \to [0, \infty)$ with $\varphi(t) = 0 \iff t = 0$ and φ is continuous at 0 we have that $\varphi \circ q \in \mathcal{P}$: Let $q \in \mathcal{P}$ and $\varphi : \operatorname{conv}(q(A)) \to [0, \infty)$ an increasing function with $\varphi(t) = 0 \iff t = 0$ and φ is continuous at 0.
There exist $n \in \mathbb{N}$ and $a_1, \dots, a_n > 0$ such that $q = a_1 p_{U_1} \vee \dots \vee a_n p_{U_n}$.
Then $\varphi \circ q = \varphi(a_1) p_{U_1} \vee \dots \vee \varphi(a_n) p_{U_n}$, for let $x \in A$.

- If $x \notin U_n$, let $i \in \{1, \dots, n-1\}$ be such that $x \in U_i$ and $x \notin U_{i+1}$. Then $q(x) = a_{i+1} \vee \dots \vee a_n$ and hence $\varphi(q(x)) = \varphi(a_{i+1} \vee \dots \vee a_n) = \varphi(a_{i+1}) \vee \dots \vee \varphi(a_n) = \varphi(a_1) p_{U_1}(x) \vee \dots \vee \varphi(a_n) p_{U_n}(x)$.
- If $x \in U_n$ then $q(x) = 0$ and hence $\varphi(q(x)) = \varphi(0) = 0$ and also $\varphi(a_1) p_{U_1}(x) \vee \dots \vee \varphi(a_n) p_{U_n}(x) = 0$.

Now $\varphi(a_1), \dots, \varphi(a_n) > 0$ for $\varphi(t) = 0 \iff t = 0$ and hence $\varphi \circ q \in \mathcal{P}$.
4. Now we show that $\mathcal{P}$ is not a strong base of seminorms. To this end let $p = p_{U_1} \vee \frac{1}{2} p_{U_2} \vee \frac{1}{3} p_{U_3} \vee \dots$. Then $p = \lim_{n \to \infty} p_n$ uniformly on A where $p_n = p_{U_1} \vee \frac{1}{2} p_{U_2} \vee \dots \vee \frac{1}{n} p_{U_n}$ for all $n \in \mathbb{N}$. Now $p_n \in \mathcal{P}$ for all $n \in \mathbb{N}$, so p is a τ-continuous seminorm.
As $\operatorname{Ker} p = \{0\}$ and $\operatorname{Ker} q \neq \{0\}$ for every $q \in \mathcal{P}$ we obtain that there do not exist a $q \in \mathcal{P}$ and a $c > 0$ such that $p \le cq$.
We see that $\mathcal{P}$ is not a strong base of seminorms.

Base of Seminorms for Locally Convex K-vector Spaces

Recall that a norm $\| \ \|$ on a K-vector space E is called a vector space seminorm if $\|\lambda x\| = |\lambda| \|x\|$ for every $\lambda \in K$ and every $x \in E$.
For a locally convex vector space (E, τ) over a non-trivially valued K the following is well-known.

If $\mathcal{P}$ is a collection of vector space seminorms on E such that

1. $\tau = \mathcal{P}$-topology,
2. $\mathcal{P}$ is closed with respect to finite suprema,
3. $cp \in \mathcal{P}$ for every $p \in \mathcal{P}$ and every $c > 0$.

Then for every τ-continuous vector space seminorm q there exists a $p \in \mathcal{P}$ such that $q \le p$.

Hence, here we may drop the condition that $\mathcal{P}$ is closed with respect to uniform limits and also that if $p \in \mathcal{P}$ and $\varphi : \operatorname{conv}(p(E)) \to [0, \infty)$ is an increasing function such that $\varphi(t) = 0 \iff t = 0$ and φ is continuous at 0, then also $\varphi \circ p \in \mathcal{P}$. (Compare Theorem 3.4.29.)

However, on the subject of uniform limits we observe the following.

If E is a vector space over a non-trivial valued K and p and q are vector space seminorms on E such that $\sup_{x,y \in E} |p(x) - p(y)| \le 1$ then $p = q$ on E.

Hence if $p_1, p_2, p_3, \ldots$ are vector space seminorms on E and $p = \lim_{n\to\infty} p_n$ uniformly on E then $p_n = p$ for large n. This means that every collection of vector space seminorms on E is closed with respect to uniform limits.

Similarly, on the functions $\varphi : \operatorname{conv}(p(E)) \to [0,\infty)$ we observe the following.

3.4.31 Proposition *Let the valuation on K be non-trivial. Let A be a B_K-module and p and q faithful seminorms on A such that there exists an increasing function $\varphi : \operatorname{conv}(p(A)) \to [0,\infty)$ with $q = \varphi \circ p$. Then:*

1. *If the valuation on K is dense there exists a $c \geq 0$ such that $q = cp$.*
2. *If the valuation on K is discrete then let $\pi = \max\{|\lambda| \mid \lambda \in B_K^-\}$. Then there exist $0 \leq c_1 \leq c_2$ with $c_1 \geq \pi c_2$ such that $c_1 p \leq q \leq c_2 p$.*

Proof: First we derive some properties for φ that are useful for the proofs of both 1. and 2..
If $q = 0$ we can take $c = c_1 = c_2 = 0$ and we are done. Henceforth we assume $q \neq 0$. Then also $p \neq 0$.
We first observe $\varphi(0) = \varphi(p(0)) = q(0) = 0$.
Let $t \in (0,\infty)$. Let $y \in A$ such that $q(y) > 0$. Then also $p(y) > 0$. There exists a $\lambda \in B_K \setminus \{0\}$ such that $|\lambda| p(y) \leq t$. Then $\varphi(t) \geq \varphi(|\lambda| p(y)) = \varphi(p(\lambda y)) = q(\lambda y) = |\lambda| q(y) > 0$.
Hence $\varphi(t) = 0 \iff t = 0$, in particular $\operatorname{Ker} p = \operatorname{Ker} q$. For $x \in A$ with $p(x) > 0$ we define $c(x) = \frac{\varphi(p(x))}{p(x)}$.
Let $x, y \in A$ such that $p(x) > p(y) > 0$. Let $\lambda \in B_K \setminus \{0\}$.
If $|\lambda| p(x) \leq p(y)$, then $|\lambda| c(x) p(x) = |\lambda| \varphi(p(x)) = \varphi(p(\lambda x)) = \varphi(|\lambda| p(x)) \leq \varphi(p(y)) = c(y) p(y)$ and hence $|\lambda| p(x) \leq \frac{c(y)}{c(x)} p(y)$.
If $|\lambda| p(x) \geq p(y)$ then in the same way $|\lambda| p(x) \geq \frac{c(y)}{c(x)} p(y)$.
Hence, if $x, y \in A$ such that $p(x) > p(y) > 0$ and $\lambda \in B_K \setminus \{0\}$, then

$$(*) \begin{cases} |\lambda| p(x) \leq p(y) \Rightarrow |\lambda| p(x) \leq \frac{c(y)}{c(x)} p(y) \\ |\lambda| p(x) \geq p(y) \Rightarrow |\lambda| p(x) \geq \frac{c(y)}{c(x)} p(y) \end{cases}$$

Now we prove 1..
Let $x, y \in A$ such that $p(x), p(y) > 0$. If $p(x) = p(y)$, then $c(x) = c(y)$. Suppose $p(x) \neq p(y)$. By symmetry we may assume that $p(x) > p(y)$. Let $0 < \varepsilon < 1$ such that $p(x) \geq (1+\varepsilon) p(y)$. As $|K|$ is dense there exist $\lambda, \mu \in B_K$ such that

$$(1-\varepsilon) p(y) \leq |\lambda| p(x) \leq p(y) \leq |\mu| p(x) \leq (1+\varepsilon) p(y).$$

By using $(*)$ we obtain that

$$(1-\varepsilon) p(y) \leq |\lambda| p(x) \leq \frac{c(y)}{c(x)} p(y) \leq |\mu| p(x) \leq (1+\varepsilon) p(y).$$

And hence

$$1 - \varepsilon \leq \frac{c(y)}{c(x)} \leq 1 + \varepsilon.$$

We obtain that $c(y) = c(x)$. Thus, by taking $c = c(x)$, we obtain $q = cp$.
Now we prove 2..
Let $x, y \in A$ such that $p(x), p(y) > 0$. By symmetry we may assume that $p(x) \geq p(y)$. Let $\lambda \in B_K$ such that $|\lambda| = \pi$. Let $n \in \mathbb{N}$ such that

$$|\lambda| p(y) \leq |\lambda^n| p(x) \leq p(y) \leq |\lambda^{n-1}| p(x) \leq |\lambda|^{-1} p(y).$$

By using $(*)$ we obtain that

$$|\lambda| p(y) \leq |\lambda^n| p(x) \leq \frac{c(y)}{c(x)} p(y) \leq |\lambda^{n-1}| p(x) \leq |\lambda|^{-1} p(y).$$

And hence

$$\pi \leq \frac{c(y)}{c(x)} \leq \pi^{-1}.$$

Let $C = \{c(x) \mid x \in A,\ p(x) > 0\}$. Let $c_1 = \inf C$ and $c_2 = \sup C$. Then $0 \leq c_1 \leq c_2$, $c_1 \geq \pi c_2$ and $c_1 p \leq q \leq c_2 p$. □

The π in 2. of this proposition is the sharpest bound as we see in the following example.

3.4.32 Example Let $(K, |\ |) = (\mathbb{Q}_3, |\ |_3)$. Then $\pi = \frac{1}{3} = \max |B_K^-|$. Let $A = B_K^{\mathbb{N}}$ and let $p : A \to [0, \infty)$ be defined by

$$p(x) = \max(|x_0|, 2|x_1|, \frac{3}{2}|x_2|, \frac{4}{3}|x_3|, \dots) \quad \left(x = (x_0, x_1, x_2, \dots) \in B_K^{\mathbb{N}}\right).$$

Then p is a faithful (semi-)norm on A and

$$p(A) = \{3^k \mid k \leq 0\} \cup \bigcup_{n \geq 1} \{(1 + \frac{1}{n})3^k \mid k \leq 0\}.$$

Let $\varphi : [0, 2] \to [0, \infty)$ be defined by

$$\varphi(t) = \begin{cases} 0 & \text{if } t = 0, \\ 3^k & \text{if } 3^{k-1} < t \leq 3^k \quad (k \leq 0), \\ 3 & \text{if } 1 < t \leq 2. \end{cases}$$

Then φ is increasing, $\varphi(t) = 0 \iff t = 0$ and φ is continuous at 0. Hence, $\varphi \circ p$ is a seminorm on A (and $\varphi \circ p \sim p$).
$\varphi \circ p$ is also faithful. For let $x \in A$ and $\lambda \in B_K$. There exists a $k \leq 0$ such that $|\lambda| = 3^k$. Let $l \leq 1$ such that $3^{l-1} < p(x) \leq 3^l$. Then $3^{k+l-1} < |\lambda| p(x) \leq 3^{k+l}$ and hence $\varphi(p(\lambda x)) = \varphi(|\lambda| p(x)) = 3^{k+l} = 3^k 3^l = |\lambda| \varphi(p(x))$.
Now $\frac{\varphi(p(e_0))}{p(e_0)} = \frac{\varphi(1)}{1} = 1$ and $\frac{\varphi(p(e_n))}{p(e_n)} = \frac{\varphi(1+\frac{1}{n})}{1+\frac{1}{n}} = \frac{3}{1+\frac{1}{n}}$ for all $n \geq 1$.
(Here e_n is the element of A with a 1 on the n^{th} place and a 0 on the other places $(n \in \mathbb{N})$.)
We see that $p(x) \leq \varphi(p(x)) \leq 3p(x) \quad (x \in A)$ and there do not exist constants c_1, c_2 with $1 < c_1 \leq c_2 < 3$ such that $c_1 p \leq \varphi \circ p \leq c_2 p$.

3.4.33 Remark Proposition 3.4.31 is true in particular for a vector space E over a non-trivial valued K and vector space seminorms p and q on E. Thus, if the valuation on K is dense and $\mathcal{P}$ is a collection of vector space seminorms on a K-vector space E that is closed with respect to multiplication with scalars > 0 then it is also true that if $p \in \mathcal{P}$ and $\varphi : [0, \infty) \to [0, \infty)$ is an increasing function such that also $\varphi \circ p$ is a vector space seminorm, then $\varphi \circ p \in \mathcal{P}$ (and $\varphi \circ p$ is a multiple of p).
This conclusion is in general not true if the valuation of K is discrete, but in that case there exists a $c > 0$ such that $cp \in \mathcal{P}$ and $\varphi \circ p \leq cp$.

3.4.34 Remark Let (E, τ) be a locally convex vector space over a non-trivial valued K and let $\mathcal{P}$ be a collection of continuous vector space seminorms on E. Then it is well-known that the following two assertions are equivalent.

1. For every continuous vector space seminorm q there exists a $p \in \mathcal{P}$ and a $c > 0$ such that $q \leq cp$.

2. The collection $(\{x \in E \mid p(x) < \varepsilon\})_{p\in\mathcal{P},\varepsilon>0}$ is a base of zero neighbourhoods.

For a locally convex B_K-module (A, τ) the counterpart of this equivalence does not hold.
For example, let $(A, \tau) = (K, |\ |)$, where $|\ |$ is non-trivial. Let $\mathcal{P} = \{|\ |\}$. Then the collection $(\{\lambda \in K \mid |\lambda| < \varepsilon\})_{\varepsilon>0}$ is a base of zero neighbourhoods. But $\sqrt{|\ |}$ is a continuous (semi-)norm and there does not exist a $c > 0$ such that $\sqrt{|\ |} \leq c|\ |$.

Other Definitions of a Base of Seminorms

The observations in Remark 3.4.34 lead to an alternative definition of a base of seminorms that is in better correspondence with a base of zero neighbourhoods for the topology.

3.4.35 Definition Let (A, τ) be a locally convex B_K-module and let $\mathcal{P}$ be a collection of seminorms on A. Then $\mathcal{P}$ is called a *weak base of seminorms* for τ if the collection $(\{x \in A \mid p(x) < \varepsilon\})_{p\in\mathcal{P},\ \varepsilon>0}$ is a base of zero neighbourhoods.

3.4.36 Proposition *Let (A, τ) be a locally convex B_K-module. Then every strong base of seminorm for τ is also a weak base of seminorms.*

This proposition will follow from Proposition 3.4.39. The following is easy to verify.

3.4.37 Proposition *Let (A, τ) be a locally convex B_K-module and let $\mathcal{P}$ be a collection of seminorms on A such that*

1. *$\tau = \mathcal{P}$-topology,*

2. *$\mathcal{P}$ is closed with respect to finite suprema.*

Then $\mathcal{P}$ is a weak base of seminorms for τ.
If, in addition,

3. $cp \in \mathcal{P}$ *for all* $p \in \mathcal{P}$ *and all* $c > 0$,

then the collection $(\{x \in A \mid p(x) < 1\})_{p\in\mathcal{P}}$ *is a base of zero neighbourhoods.*

Somewhere in between strong and weak base of seminorms is the following notion.

3.4.38 Definition A collection $\mathcal{P}$ of continuous seminorms on a locally convex B_K-module (A, τ) is called a $*$*-base of seminorms* if for every continuous seminorm q on A there exists a $p \in \mathcal{P}$ such that q is p-continuous.

3.4.39 Proposition *Let (A, τ) be a locally convex B_K-module. Then every strong base of seminorms is a $*$-base and every $*$-base of seminorms is a weak base.*

Proof: The first part of this proposition can be seen as follows. Let $\mathcal{P}$ be a strong base of seminorms and let p be a τ-continuous seminorm on A. If p is bounded then, by 2. of Definition 3.4.27, there exists a $q \in \mathcal{P}$ and a $c > 0$ such that $p \leq cq$. By using Lemma 3.3.17 we obtain that p is q-continuous. If p is not bounded, then $p \wedge 1$ is a bounded τ-continuous seminorm on A and hence there exist a $q \in \mathcal{P}$ and a $c > 0$ such that $(p \wedge 1) \leq cq$. Let $\varphi : [0, \infty) \to [0, \infty)$ be defined by

$$\psi(t) = \begin{cases} \frac{1}{c}t & \text{if } t < 1, \\ \frac{1}{c} & \text{if } t \geq 1, \end{cases}$$

then $\psi \circ p = \frac{1}{c}(p \wedge 1) \leq q$. Again by using Lemma 3.3.17 we obtain that p is q-continuous.
We see that $\mathcal{P}$ is a $*$-base of seminorms.
The second part of this proposition can be proved as follows. Let $\mathcal{P}$ be a $*$-base of seminorms. Let U be a zero neighbourhood of A. There exist an $n \in \mathbb{N}$, τ-continuous seminorms $p_1, \ldots, p_n$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that $\{x \in A \mid p_1(x) < \varepsilon_1, \ldots, p_n(x) < \varepsilon_n\} \subset U$. Let $q = \frac{1}{\varepsilon_1}p_1 \vee \ldots \vee \frac{1}{\varepsilon_n}p_n$. Then q is a τ-continuous seminorm and as $\mathcal{P}$ is a $*$-base there exists an $r \in \mathcal{P}$ such that q is r-continuous. From Proposition 3.3.10 we obtain that there exists a $\delta > 0$ such that $\{x \in A \mid r(x) < \delta\} \subset \{x \in A \mid q(x) < 1\}$. Then also $\{x \in A \mid r(x) < \delta\} \subset \{x \in A \mid p_1(x) < \varepsilon_1, \ldots, p_n(x) < \varepsilon_n\} \subset U$.
We see that $\mathcal{P}$ is a weak base of seminorms. □

3.4.40 Remark Recall that in Theorem 3.4.29 we have seen the following. Let $\mathcal{P}$ a collection of seminorms on a locally convex B_K-module (A, τ) such that

1. $\tau = \mathcal{P}$-topology,

2. $\mathcal{P}$ is closed with respect to finite suprema and uniform limits,

3. if $p \in \mathcal{P}$ and $\varphi : \operatorname{conv}(p(A)) \to [0,\infty)$ is an increasing map such that $\varphi(t) = 0 \iff t = 0$ and φ is continuous at 0, then also $\varphi \circ p \in \mathcal{P}$.

Then $\mathcal{P}$ is a strong base of seminorms and hence also a $*$-base of seminorms.
We want to weaken these conditions 1., 2., 3. in such a way that they still imply that $\mathcal{P}$ is a $*$-base of seminorms.
The condition on the closedness with respect to uniform limits we cannot omit. For this consider again the second example of Remark 3.4.30. Then $p = p_{U_1} \vee \frac{1}{2}p_{U_2} \vee \frac{1}{3}p_{U_3} \vee \dots$ is a continuous seminorm.
Suppose there exists a $q \in \mathcal{P}$ such that p is q-continuous. There exist $n \in \mathbb{N}$ and $a_1, \dots, a_n > 0$ such that $q = a_1 p_{U_1} \vee \dots \vee a_n p_{U_n}$. Then $\operatorname{Ker} p = \{0\}$ whereas $\operatorname{Ker} q = U_n$. This is in contradiction with Proposition 3.3.11.
Hence, $\mathcal{P}$ is not a $*$-base of seminorms for τ.

Condition 3. can be weakened in the following way.

3.4.41 Theorem *Let (A,τ) be a locally convex B_K-module and let $\mathcal{P}$ be a collection of bounded seminorms on A such that*

1. *$\tau = \mathcal{P}$-topology,*

2. *$\mathcal{P}$ is closed with respect to finite suprema and uniform limits,*

3. *if $p \in \mathcal{P}$ and $c > 0$ then $cp \in \mathcal{P}$.*

Then $\mathcal{P}$ is a $*$-base of seminorms.

Proof: Let q be a continuous seminorm. The set $V_n = \{x \in A \mid q(x) < \frac{1}{n}\}$ is a $\mathcal{P}$-open submodule for every $n \geq 1$. Hence, according to 2. and 3., for all $n \geq 1$ there exists a $p_n \in \mathcal{P}$ such that $\{x \in A \mid p_n(x) < 1\} \subset V_n$. As each p_n is bounded there exist $c_1 > c_2 > c_3 > \dots$ with $\lim_{n\to\infty} c_n = 0$ such that $c_n p_n < \frac{1}{n}$ for all $n \geq 1$. Let

$$p = c_1 p_1 \vee c_2 p_2 \vee c_3 p_3 \vee \dots .$$

Then $p = \lim_{n\to\infty} \tilde{p}_n$ uniformly on A, where $\tilde{p}_n = c_1 p_1 \vee \dots \vee c_n p_n$ for all $n \geq 1$. As $\tilde{p}_n \in \mathcal{P}$ for all $n \in \mathbb{N}$ we obtain $p \in \mathcal{P}$. Let $\varphi : [0,\infty) \to [0,\infty)$ be defined by

$$\varphi(t) = \begin{cases} 0 & \text{if } t = 0, \\ c_n & \text{if } \frac{1}{n} \leq t < \frac{1}{n-1} \quad (n \geq 2), \\ c_1 & \text{if } t \geq 1. \end{cases}$$

Then φ is increasing and $\varphi(t) = 0 \iff t = 0$. Furthermore, it is not hard to verify that $\varphi \circ q \leq p$. By Lemma 3.3.17, q is p-continuous. □

A drawback of this theorem is that the seminorms in $\mathcal{P}$ must be bounded and this condition can not be omitted as we see in the following example.

3.4.42 Example Let the valuation on K be non-trivial. Let τ be the $|\ |$-topology on K. For every $n \geq 1$ let $U_n = \{\lambda \in K \mid |\lambda| < \frac{1}{n}\}$ and let the seminorm p_n be defined by

$$p_n(\lambda) = \begin{cases} 0 & \text{if } \lambda \in U_n, \\ 1 & \text{if } \lambda \in U_1 \setminus U_n, \\ |\lambda|^n & \text{if } |\lambda| \geq 1. \end{cases}$$

Let $\mathcal{Q} = \{a_1p_1 \vee a_2p_2 \vee \ldots \vee a_np_n \mid n \in \mathbb{N}, a_1, \ldots, a_n > 0\}$. It is not hard to see that every $q \in \mathcal{Q}$ is τ-continuous. Furthermore, $\mathcal{Q}$ generates τ. For let U be a τ-open submodule of K. Then there exists an $n \in \mathbb{N}$ such that $U_n \subset U$. Let $q = p_1 \vee \ldots \vee p_n$, then $q \in \mathcal{Q}$ and $\{\lambda \in K \mid q(\lambda) < 1\} = U_n \subset U$. It is not hard to see that $\mathcal{Q}$ is closed with respect to finite suprema and multiplication by positive numbers . Let $\mathcal{P}$ be the collection of all seminorms on K that are uniform limits of a sequence in $\mathcal{Q}$. Then every $q \in \mathcal{P}$ is τ-continuous and as $\mathcal{Q} \subset \mathcal{P}$ it follows that the $\mathcal{P}$-topology equals τ. Now $\mathcal{P}$ is closed with respect to uniform limits. Moreover, $\mathcal{P}$ is closed with respect to finite suprema, for if $r, r' \in \mathcal{P}$ then there exist sequences $(r_n)_{n\in\mathbb{N}}$ and $(r'_n)_{n\in\mathbb{N}}$ in $\mathcal{Q}$ such that $r_n \to r$ and $r'_n \to r'$ uniformly on K. Then $r_n \vee r'_n \in \mathcal{Q}$ for every $n \in \mathbb{N}$ and $r_n \vee r'_n \to r \vee r'$ uniformly on K. In the same way we see that $\mathcal{P}$ is closed with respect to multiplication by numbers > 0.
We now prove that $\operatorname{Ker} q \neq \{0\}$ for every $q \in \mathcal{P}$. Let $q \in \mathcal{P}$. There exists a sequence $(q_n)_{n\in\mathbb{N}}$ in $\mathcal{Q}$ such that $q_n \to q$ uniformly on K. There exists an $N \in \mathbb{N}$ such that $\sup_{x,y\in A} |q_n(x) - q_m(y)| < 1$ for all $n, m \geq N$. Let $a_1, a_2, \ldots, a_s > 0$ such that $q_N = a_1p_1 \vee a_2p_2 \vee \ldots \vee a_sp_s$. Let $n \geq N$. Let $b_1, \ldots, b_k > 0$ such that $q_n = b_1p_1 \vee \ldots \vee b_kp_k$. Suppose that $k > s$. Then for all $\lambda \in K$ such that $|\lambda| \geq \max(1, \frac{a_1}{a_s}, \ldots, \frac{a_{s-1}}{a_s}, \frac{b_1}{b_k}, \ldots, \frac{b_{k-1}}{b_k})$ we have $q_n(\lambda) - q_N(\lambda) = b_k|\lambda|^k - a_s|\lambda|^s = |\lambda|^s(b_k|\lambda|^{k-s} - a_s)$. As $b_k|\lambda|^{k-s} - a_s > 1$ for large λ we obtain that $q_n(\lambda) - q_N(\lambda) \geq |\lambda|^s > 1$ for large λ, a contradiction.
Hence, $k \leq s$. We see that $U_s \subset \operatorname{Ker} q_n$ for all $n \geq N$ and as $q_n \to q$ we obtain that $U_s \subset \operatorname{Ker} q$ and hence $\operatorname{Ker} q$ is non-trivial.
Now $|\ |$ is, of course, a τ-continuous seminorm, with $\operatorname{Ker} |\ | = \{0\}$. By using Proposition 3.3.11 we obtain that $|\ |$ is q-continuous for no $q \in \mathcal{P}$. It follows that $\mathcal{P}$ is not a $*$-base of seminorms.
Note that the fact that we take the collection of all seminorms that are a uniform limit of seminorms in $\mathcal{Q}$ is a trick to shorten the example. In fact one can prove that $\mathcal{Q}$ itself is already closed with respect to uniform limits. But the proof is long and not very interesting.

Chapter 4

More on Locally Convex B_K-modules

In this chapter we will develop some general theory on locally convex B_K-modules. We are not aiming at a systematic treatment. Most subjects in this chapter will be useful for the theory of compactoids in the next chapter.

4.1 Locally Convex Topologies on B_K-modules of Finite Rank

In section 3.3 of the previous chapter we have proved that each two norms on a torsion free B_K-module of finite rank are equivalent. Here we will prove that on such a module there exists only one Hausdorff locally convex topology.

4.1.1 Proposition *Let $E = Ke$ be a one-dimensional K-vector space. Let the norm ν on E be defined by $\nu(\lambda e) = |\lambda|$ $(\lambda \in B_K)$ Then the ν-topology is the only locally convex Hausdorff (module) topology on E.*

Proof: Let τ be a locally convex Hausdorff topology on E. If $|K|$ is trivial then there exists a τ-open submodule U of E such that $e \notin U$. This implies $U = \{0\}$ and hence τ is discrete. Now ν is a trivial norm on E and hence τ equals the ν-topology.
Now suppose $|K|$ is non-trivial. Let $(\lambda_\alpha e)_{\alpha\in I}$ be a net in E. We prove

$$\lambda_\alpha e \xrightarrow{\tau} 0 \iff |\lambda_\alpha| \to 0.$$

Suppose $\lambda_\alpha e \xrightarrow{\tau} 0$. Let $\varepsilon > 0$. Let $\mu \in B_K \setminus \{0\}$ such that $|\mu| < \varepsilon$. As τ is Hausdorff there exists a τ-continuous seminorm p such that $p(\mu e) = 1$. Now $\lambda_\alpha e \xrightarrow{\tau} 0$ and hence $p(\lambda_\alpha e) < 1$ for large α. This implies $|\lambda_\alpha| < |\mu| < \varepsilon$ for large α. Thus, $|\lambda_\alpha| \to 0$.

Suppose $|\lambda_\alpha| \to 0$. Then $|\lambda_\alpha| \leq 1$ for large α and since the scalar multiplication $B_K \times E \to E$ is continuous it follows that $\lambda_\alpha e \xrightarrow{\tau} 0$. □

4.1.2 Proposition *Let E be a finite-dimensional K-vector space. Then there exists exactly one locally convex Hausdorff (module) topology τ on E. Moreover, τ is normable and (E, τ) is complete.*

Proof: By induction to $n = \dim E$. The case $n = 1$ is the previous proposition.
Let E be an n-dimensional vector space. Let $e_1, \dots, e_n$ be a (vector space) base for E. Then $E = [e_1] \oplus [e_2, \dots, e_n]$. There exists a norm $\| \ \|_1$ on $[e_1]$ such that τ equals the $\| \ \|_1$-topology on $[e_1]$ and a norm $\| \ \|_2$ on $[e_2, \dots, e_n]$ such that τ equals the $\| \ \|_2$-topology on $[e_2, \dots, e_n]$. Let $\| \ \|$ on E be defined in the following way. Let $x \in E$. Let $y \in [e_1]$ and $z \in [e_2, \dots, e_n]$ such that $x = y + z$. Then $\|x\| = \|y\|_1 \vee \|z\|_2$. Then $\| \ \|$ is a norm on E. Let $(x_\alpha)_{\alpha \in I}$ be a net in E. We prove

$$x_\alpha \xrightarrow{\tau} 0 \iff \|x_\alpha\| \to 0.$$

Let $(y_\alpha)_{\alpha \in I}$ be a net in $[e_1]$ and let $(z_\alpha)_{\alpha \in I}$ be a net in $[e_2, \dots, e_n]$ such that $x_\alpha = y_\alpha + z_\alpha$ for all $\alpha \in I$.
Suppose $x_\alpha \xrightarrow{\tau} 0$. Assume that not $\|y_\alpha\| \to 0$. Let $(\lambda_\alpha)_{\alpha \in I}$ be a net in K such that $y_\alpha = \lambda_\alpha e_1$ for all $i \in I$. Then not $\lambda_\alpha \to 0$. We may suppose (if necessary by taking a suitable cofinal subset of I) that there exists a $\mu \in B_K \setminus \{0\}$ such that $|\lambda_\alpha| \geq |\mu|$ for all $\alpha \in I$. Let $\mu_\alpha = \mu\lambda_\alpha^{-1}$ $(\alpha \in I)$. Then $(\mu_\alpha)_{\alpha \in I}$ is a net in B_K and hence $\mu_\alpha x_\alpha = \mu e_1 + \mu_\alpha z_\alpha \xrightarrow{\tau} 0$. Thus, $\mu_\alpha z_\alpha \xrightarrow{\tau} -\mu e_1$. As $[e_2, \dots, e_n]$ is complete and hence closed with respect to τ we obtain that $-\mu e_1 \in [e_2, \dots, e_n]$, a contradiction.
Thus, $\lambda_\alpha \to 0$, which implies that $\|\lambda_\alpha e_1\|_1 \to 0$ and hence $y_\alpha \xrightarrow{\tau} 0$. Then also $z_\alpha = x_\alpha - y_\alpha \xrightarrow{\tau} 0$ and hence also $\|z_\alpha\|_2 \to 0$.
We obtain that $\|x_\alpha\| \to 0$.
Suppose that $\|x_\alpha\| \to 0$. Then $\|y_\alpha\|_1 \to 0$ and $\|z_\alpha\|_2 \to 0$. Then $y_\alpha \xrightarrow{\tau} 0$ and $z_\alpha \xrightarrow{\tau} 0$. Then also $x_\alpha \xrightarrow{\tau} 0$.
Wee see that $\tau = \| \ \|$-topology and from Theorem 3.3.23 we obtain that (E, τ) is complete. □

4.1.3 Proposition *Let A be a B_K-module such that there exists exactly one locally convex Hausdorff topology on A. Let B be a submodule of A. Then there exists also exactly one locally convex Hausdorff topology on B.*

Proof: Let τ and σ be locally convex Hausdorff topologies on B. From Proposition 3.4.23 we obtain that there exist locally convex Hausdorff topologies τ' and σ' on A such that $\tau'|B = \tau$ and $\sigma'|B = \sigma$. Then $\tau' = \sigma'$ and hence $\tau = \sigma$. □

4.1.4 Proposition *Let A be a torsion free B_K-module of finite rank. Then there exists only one locally convex Hausdorff topology τ on A. Moreover, this topology is normable and complete.*

Proof: $A \subset K \otimes_{B_K} A$ and the latter one is a finite-dimensional K-vector space (see Remark 2.2.29). Combining Proposition 4.1.2 and Proposition 4.1.3 we obtain that there exists a unique locally convex Hausdorff topology τ on A and τ is normable. From Theorem 3.3.25 it follows that (A, τ) is complete. □

The following is a generalization of Proposition 3.3.26.

4.1.5 Proposition *Let $|K|$ be non-trivial. Let A, B be absolutely convex subsets of K such that $\{0\} \subsetneq B \subsetneq A$. Let $V = \operatorname{conv}|B|$. Let τ be a locally convex Hausdorff topology on A/B.*

1. *If $|K|$ is discrete or $|K|$ is dense and $V = [0, c)$ with $c \in |K|$ then τ is discrete.*

2. *If $|K|$ is dense and $V = [0, c]$ with $c \in |K|$ or $V = [0, c)$ with $c \notin |K|$ then either τ is discrete or τ equals the $\| \ \|$-topology, where $\| \ \|$ is defined by $\|\lambda + B\| = (|\lambda| - c) \vee 0 \quad (\lambda \in A)$.*

Proof: 1. We define $d = \inf\{|\lambda| \mid |\lambda| \notin V\}$. Then $d \in |K|$. Let $\mu \in K$ such that $|\mu| = d$. Then $\mu \in A \setminus B$.
Let $(x_\alpha)_{\alpha \in I}$ be a net in A/B and suppose not $x_\alpha = 0$ for large α. Let $J = \{\beta \in I \mid x_\beta \neq 0\}$. Then J is a cofinal subset of I. Let $(\lambda_\beta)_{\beta \in J}$ be a net in A such that $x_\beta = \lambda_\beta + B \quad (\beta \in J)$. Then $|\lambda_\beta| \notin V$ and hence $|\lambda_\beta| \geq |\mu|$ for all $\beta \in J$. Then $(\mu\lambda_\beta^{-1})_{\beta \in J}$ is a net in B_K and $\mu\lambda_\beta^{-1} x_\beta = \mu + B \quad (\beta \in J)$. As τ is Hausdorff we obtain that not $\mu\lambda_\beta^{-1} x_\beta \xrightarrow{\tau} 0$. From Proposition 3.1.29 we obtain that then also not $x_\alpha \xrightarrow{\tau} 0$.
Hence, if $(x_\alpha)_{\alpha \in I}$ is a net in A/B such that $x_\alpha \xrightarrow{\tau} 0$ then $x_\alpha = 0$ for large α. This means that τ is discrete.
2. Suppose τ is not discrete. Let $(x_\alpha)_{\alpha \in I}$ be a net in A/B. Let $(\lambda_\alpha)_{\alpha \in I}$ be a net in A such that $x_\alpha = \lambda_\alpha + B \quad (\alpha \in I)$. We prove

$$x_\alpha \xrightarrow{\tau} 0 \iff \|x_\alpha\| \to 0.$$

Suppose $x_\alpha \xrightarrow{\tau} 0$. Let $\varepsilon > 0$. Let $\mu \in A$ such that $c < |\mu| < c + \varepsilon$. Then $\mu + B \neq 0$ and as τ is Hausdorff there exists a τ-open submodule U of A/B with $\mu + B \notin U$. Now $x_\alpha \in U$ for large α and hence $|\lambda_\alpha| < |\mu| < c + \varepsilon$ for large α. This implies that $\|x_\alpha\| < \varepsilon$ for large α. We obtain $\|x_\alpha\| \to 0$.
Suppose $\|x_\alpha\| \to 0$. Let U be a τ-open submodule. As τ is not discrete there exists a $y \in U$ with $y \neq 0$. Let $\mu \in A$ such that $y = \mu + B$. Then $|\mu| > c$. As $\|x_\alpha\| \to 0$ we obtain that $|\lambda_\alpha| < |\mu|$ for large α and hence $x_\alpha = \lambda_\alpha + B \in U$ for large α. We obtain that $x_\alpha \xrightarrow{\tau} 0$.
We see that τ is discrete or τ equals the $\| \ \|$-topology. □

The following we will need in the proof of Theorem 5.1.6 in the next chapter.

4.1.6 Corollary *Let A be a B_K-module of rank 1. Then every Hausdorff locally convex topology on A is normable.*

Proof: If A is torsion free the statement follows from Proposition 4.1.4. If A is a torsion module and τ is not discrete the statement follows from the previous proposition. If τ is discrete then $\| \ \|$ defined by

$$\|x\| = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \neq 0 \end{cases}$$

is a trivial norm on A and hence $\tau = \| \ \|$-topology. □

We do not know whether this corollary also holds for B_K-modules of rank n with $n > 1$.

4.1.7 Proposition *Let $A \in \mathcal{F}_K$ and let* $\operatorname{rank} A = n$. *Let τ be a locally convex topology on A. Let B be an absolutely convex subset of K^n and let $\varphi : B \to A$ be a surjective homomorphism. Let σ be the unique locally convex Hausdorff topology on B. Then $\varphi : (B, \sigma) \to (A, \tau)$ is continuous.*

Proof: Let U be an open submodule of A. Let $x \in K^n$. By Proposition 2.2.28 there exists a $\lambda \in B_K \setminus \{0\}$ such that $\lambda x \in B$. As U is absorbing in A there exists a $\mu \in B_K \setminus \{0\}$ such that $\mu\varphi(\lambda x) \in U$. Then $\varphi(\mu\lambda x) = \mu\varphi(\lambda x) \in U$ and hence $\mu\lambda x \in \varphi^{-1}(U)$. We obtain that $\varphi^{-1}(U)$ is an absorbing submodule of B and hence open. □

4.1.8 Proposition *Let K be spherically complete. Let $A \in \mathcal{F}_K$ and let τ be a locally convex Hausdorff topology on A. Then (A, τ) is complete and every submodule of A is closed.*

Proof: Let B be a submodule of A. Let $(x_\alpha)_{\alpha \in I}$ be a Cauchy net in $(B, \tau | B)$. For $\alpha \in I$ let the set C_α be defined by

$$\{\lambda_1 x_{\beta_1} + \cdots + \lambda_n x_{\beta_n} \mid n \in \mathbb{N},\ \beta_1, \ldots, \beta_n \succ \alpha,\ \lambda_1, \ldots, \lambda_n \in B_K,\ \sum_{i=1}^{n} \lambda_n = 1\}.$$

(That is to say C_α is the convex hull of $\{x_\beta \mid \beta \succ \alpha\}$.) Then $(C_\alpha)_{\alpha \in I}$ is a collection of non-empty convex subsets of B with the finite intersection property. By Theorem 2.3.22 B is linearly compact and hence there exists an $x \in B$ such that $x \in \bigcap_{\alpha \in I} C_\alpha$. Suppose not $x_\alpha \to x$. Then there exists a convex neighbourhood U of x such that for every $\alpha \in I$ there exists a $\beta \succ \alpha$ such that $x_\beta \notin U$. Let $\alpha \in I$ be such that $x_\gamma - x_\beta \in U - \{x\}$ for every $\beta, \gamma \succ \alpha$. Let $\beta \succ \alpha$ be such that $x_\beta \notin U$. Let $V = U - \{x\} + \{x_\beta\}$. Then V is convex and $x_\gamma \in V$ for all $\gamma \succ \beta$, hence $C_\beta \subset V$, and therefore also $x \in V$. Then $x = u - x + x_\beta$ for some $u \in U$. Then $x_\beta = x + x - u$ and $x + x - u \in U$, for U is convex. This is in contradiction with the choice of β. Hence, $x_\alpha \to x$. We see that every submodule of A is complete. In particular (A, τ) is complete and as τ is Hausdorff every submodule of A is closed. □

4.1.9 Remark Let K be not spherically complete. In Example 3.2.12 we have seen a normed B_K-module $B = (B_K / \mathrm{B}(0, r), \| \ \|)$ that is not complete. Let $\widehat{B}$ be the completion of B and let $e \in \widehat{B} \setminus B$. Then $B + \operatorname{co}\{e\}$ is a B_K-module of finite rank and B is a submodule of rank 1 of $B + \operatorname{co}\{e\}$ that is not closed.

The Strongest Locally Convex Topology

The following definition is suitable for arbitrary B_K-modules, not only for B_K-modules of finite rank.

4.1.10 Definition Let A be a B_K-module. The *strongest locally convex topology* on A is the discrete topology if $|K|$ is trivial and, if $|K|$ is non-trivial, it is the topology generated by all the absorbing submodules of A. The strongest locally convex topology on A is denoted τ_A.

4.1.11 Lemma *Let A be a B_K-module. Then*

1. *Every seminorm on A is τ_A-continuous.*
2. *τ_A is Hausdorff.*
3. *If B is a submodule of A then $\tau_B = \tau_A|B$.*
4. *Every submodule of A is τ_A-closed.*
5. *Let B be a submodule of A. Then $\tau_{A/B}$ equals the quotient topology of τ_A.*

Proof: If $|K|$ is trivial the lemma is obvious. Hence we suppose $|K|$ is non-trivial.
1. Let p be a seminorm on A. Then $\{x \in A \mid p(x) < \varepsilon\}$ is an absorbing submodule for every $\varepsilon > 0$. By Proposition 3.3.7 p is τ_A-continuous.
2. Let $x \in A$, $x \neq 0$. Let the seminorm p on $\operatorname{co}\{x\}$ be defined by

$$p(\lambda x) = \begin{cases} 0 & \text{if } |\lambda| < 1, \\ 1 & \text{if } |\lambda| = 1. \end{cases}$$

By Theorem 3.3.4 we obtain that there exists a seminorm q on A with $q|\operatorname{co}\{x\} = p$. By 1. q is τ_A-continuous and $q(x) = 1$.
3. Let p be a seminorm on B. By Proposition 3.3.16 there exists a bounded seminorm q on B such that $p \sim q$. By Theorem 3.3.4 there exists a seminorm r on A such that $r|B = q$. This r is τ_A-continuous and hence $q = r|B$ is $\tau_A|B$-continuous and as $p \sim q$ also p is $\tau_A|B$-continuous. We see that every seminorm on B is $\tau_A|B$-continuous and hence $\tau_A|B = \tau_B$.
4. Let B be a submodule of A. If $B = A$ then B is closed. Suppose $B \neq A$. Let $x \in A \setminus B$. Let p on $B + \operatorname{co}\{x\}$ be defined by

$$p(b + \lambda x) = \begin{cases} 0 & \text{if } |\lambda| < 1, \\ 1 & \text{if } |\lambda| = 1, \end{cases}$$

($b \in B$, $\lambda \in B_K$). In the same way as in the proof of Theorem 3.4.22 we obtain that p is a well-defined seminorm on $B + \operatorname{co}\{x\}$. From Theorem 3.3.4 we obtain that there exists a seminorm q on A such that $q|B + \operatorname{co}\{x\} = p$. Let $V = \{y \in A \mid q(y) < 1\}$. Then $x + V$ is τ_A-open and $(x + V) \cap B = \emptyset$. We conclude that B is closed.
5. Let $\pi : A \to A/B$ be the quotient map. Let U be an absorbing submodule in A/B. Then $\pi^{-1}(U)$ is an absorbing submodule of A. Hence, $\pi^{-1}(U)$ is

τ_A-open. Then $U = \pi(\pi^{-1}(U))$ is open in the quotient topology of τ_A. We see that every absorbing submodule of A/B is open in the quotient topology of τ_A and hence this topology equals $\tau_{A/B}$. □

4.1.12 Proposition *Let $A \in \mathcal{F}_K$. Then (A, τ_A) is a complete normable B_K-module.*

Proof: We may assume that $A = B/\operatorname{Ker}\varphi$, where B is an absolutely convex subset of some K^n and $\varphi : B \to A$ is a surjective homomorphism. From Proposition 4.1.4 we obtain that τ_B is normable and that (B, τ_B) is complete. By Lemma 4.1.11 the quotient topology of τ_B equals τ_A. By using Proposition 3.2.17 we obtain that (A, τ_A) is normable and complete. □

4.2 B_K-modules of Countable Type

The content of this section can also be found in [18]. Our goal is to prove that a submodule of a locally convex B_K-module of countable is again of countable type, see Theorem 4.2.14
For this we need some algebra.

Preliminaries: Countably Generated B_K-modules

Recall from Definition 2.1.6 that a B_K-module A is countably generated if there exists a countable subset X of A such that $A = \operatorname{co} X$.
A K-vector space E is called a *countably generated K-vector space* if there exists a countable subset X of E such that $E = [X]$.

4.2.1 Proposition *Let E be a K-vector space. Then E is a countably generated K-vector space $\iff$ E is countably generated as a B_K-module.*

Proof: If the valuation on K is trivial there is nothing to prove, so suppose that $|\ |$ is non-trivial.
$\Rightarrow$) Let X be a countable subset of E such that $[X] = E$. Let $\lambda \in K$ such that $|\lambda| > 1$. Let $X_n = \lambda^n X := \{\lambda^n x \mid x \in X\}$ $(n \in \mathbb{N})$. Then $\bigcup_{n\in\mathbb{N}} X_n$ is a countable subset of E and $\operatorname{co} \bigcup_{n\in\mathbb{N}} X_n = [X] = E$. Hence, E is countably generated as a B_K-module.
$\Leftarrow$) is obvious. □

4.2.2 Lemma *Each B_K-submodule of a countably generated K-vector space is countably generated.*

Proof: Let E be a countably generated K-vector space and let A be a B_K-submodule of E. Let $e_1, e_2, e_3, \ldots \in E$ such that $E = [e_1, e_2, e_3, \ldots]$. For every $n \geq 1$ let $V_n = A \cap [e_1, \ldots, e_n]$. Then $A = \bigcup_{n\geq 1} V_n$. Now $V_n \in \mathcal{F}_K$ for all $n \geq 1$ and by Proposition 2.2.44 each V_n is countably generated. Then also A is countably generated. □

4.2.3 Proposition *Let A be a countably generated torsion free B_K-module. Let B be a submodule of A. Then B is also countably generated.*

Proof: It is not hard to prove that $K \otimes_{B_K} A$ is a countably generated K-vector space. As A is torsion free we obtain that A is embeddable in $K \otimes_{B_K} A$ and hence so is B. From the previous lemma we obtain that B is countably generated. □

4.2.4 Proposition *Let A be a countably generated B_K-module and let B be a submodule of A. Then B is also countably generated.*

Proof: The B_K-module

$$B_K^{(\mathrm{N})} := \{(\lambda_1, \lambda_2, \lambda_3, \dots) \mid \lambda_n \in B_K \quad (n \geq 1) \text{ and } \lambda_n = 0 \text{ for large } n\}$$

is torsion free and countably generated. Let $x_1, x_2, x_3, \dots \in A$ such that $A = \operatorname{co}\{x_1, x_2, x_3, \dots\}$. Let $h : B_K^{(\mathrm{N})} \to A$ be defined by

$$h(\lambda_1, \lambda_2, \lambda_3, \dots) = \sum_{i=1}^{\infty} \lambda_i x_i \quad (\lambda_1, \lambda_2, \lambda_3, \dots) \in B_K^{(\mathrm{N})}.$$

Then h is a surjective homomorphism. Now $h^{-1}(B)$ is a submodule of $B_K^{(\mathrm{N})}$ and hence $h^{-1}(B)$ is countably generated. Then also $B = h(h^{-1}(B))$ is countably generated. □

B_K-modules of Countable Type

In p-adic Functional Analysis Banach spaces of countable type take over the role played by separable spaces in classical theory. (Recall that a K-Banach space E is called of countable type if there exists a countable subset X of E such that $[X]$ is dense in B.) We now introduce the corresponding notion for normed B_K-modules.

4.2.5 Definition A normed B_K-module $(A, \|\ \|)$ is called *of countable type* if there exist a countable subset X of A such that $\operatorname{co} X$ is dense in A.

4.2.6 Proposition *Let $(E, \|\ \|)$ be a K-Banach space. Then $(E, \|\ \|)$ is a Banach space of countable type $\iff$ $(E, \|\ \|)$ is a normed B_K-module of countable type.*

The proof of this proposition is similar to that of Proposition 4.2.1 and will be omitted.

4.2.7 Theorem *Let $(A, \|\ \|)$ be a B_K-module of countable type. Let B be a submodule of A and let $\|\ \|_B$ be the restriction of $\|\ \|$ to B. Then also $(B, \|\ \|_B)$ is of countable type.*

Proof: For every $n \geq 1$ let $U_n = \{x \in A \mid \|x\| < \frac{1}{n}\}$. As A is of countable type we obtain that A/U_n is countably generated for every $n \geq 1$. For every $n \geq 1$ let $\pi_n : A \to A/U_n$ be the quotient map. Then $\pi_n(B)$ is countably generated for each $n \geq 1$. Let $\hat{e}_1^n, \hat{e}_2^n, \hat{e}_3^n, \dots \in \pi_n(B)$ be such that $\pi_n(B) =$

$\mathrm{co}\,\{\hat{e}_1^n, \hat{e}_2^n, \hat{e}_3^n, \ldots\}$ $(n \geq 1)$. For each $n \geq 1$ choose $e_1^n, e_2^n, e_3^n, \ldots \in B$ such that $\pi(e_i^n) = \hat{e}_i^n$ $(i \geq 1)$. We prove that $\mathrm{co}\,\{e_i^n \mid i \geq 1,\ n \geq 1\}$ is dense in B. To this end let $y \in B$ and let $\varepsilon > 0$. Let $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. There exist $m \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_m \in B_K$ such that $\pi_n(y) = \lambda_1 \hat{e}_1^n + \cdots + \lambda_m \hat{e}_m^n = \pi_n(\lambda_1 e_1^n + \cdots \lambda_n e_m^n)$. And hence $\|y - (\lambda_1 e_1^n + \cdots + \lambda_m e_m^n)\| < \frac{1}{n} < \varepsilon$. □

4.2.8 Proposition *Let $(A, \|\ \|)$ be a normed B_K-module of countable type. Let $(B, \|\ \|')$ be a normed B_K-module and $\varphi : A \to B$ a continuous, surjective homomorphism. Then also $(B, \|\ \|')$ is of countable type.*

Proof: Let X be a countable subset of A such that $\mathrm{co}\,X$ is dense in $(A, \|\ \|)$. Then $\varphi(X)$ is a countable subset of B and $\mathrm{co}\,(\varphi(X)) = \varphi(\mathrm{co}\,X)$. By Proposition 3.1.6 φ is uniformly continuous and hence, $\varphi(\mathrm{co}\,X)$ is dense in $(B, \|\ \|')$.
We conclude that $(B, \|\ \|')$ is of countable type. □

We recall the following from Proposition 3.4.16. Let A be a B_K-module and let p be a seminorm on A. Then the norm $\overline{p}$ on $A/\mathrm{Ker}\,p$ is defined by $\overline{p}(x + \mathrm{Ker}\,p) = p(x)$ $(x \in A)$.

4.2.9 Definition A locally convex B_K-module (A, τ) is called *of countable type* if for every continuous seminorm p on A the normed B_K-module $(A/\mathrm{Ker}\,p, \overline{p})$ is of countable type.

4.2.10 Proposition *A continuous homomorphic image of a locally convex B_K-module of countable type is again of countable type.*

Proof: Let (A, τ) be a locally convex B_K-module of countable type. Let (B, σ) be a locally convex B_K-module and let $\varphi : A \to B$ be a continuous surjective homomorphism. Let p be a continuous seminorm on B. We prove that $(B/\mathrm{Ker}\,p, \overline{p})$ is of countable type. In fact, $p \circ \varphi$ is a continuous seminorm on A and φ induces a continuous surjective homomorphism $(A/\mathrm{Ker}\,(p \circ \varphi), \overline{p \circ \varphi}) \to (B/\mathrm{Ker}\,\overline{p})$. Now $(A/\mathrm{Ker}\,(p \circ \varphi), \overline{p \circ \varphi})$ is a normed module of countable type and hence, by Proposition 4.2.8, also $(B/\mathrm{Ker}\,p, \overline{p})$ is of countable type.
We see that (B, σ) is of countable type. □

4.2.11 Definition Let A be a B_K-module and let p be a seminorm on A. Then p is called *of countable type* if the locally convex space (A, p) is of countable type.

4.2.12 Proposition *Let A be a B_K-module and let p be a seminorm on A. Then:*
p is of countable type $\Longleftrightarrow$ $(A/\mathrm{Ker}\,p, \overline{p})$ is of countable type.

Proof: $\Rightarrow$) (A, p) is of countable type and p is a p-continuous seminorm, hence $(A/\mathrm{Ker}\,p, \overline{p})$ is of countable type.
$\Leftarrow$) Let q be a p-continuous seminorm. We have to prove that $(A/\mathrm{Ker}\,q, \overline{q})$ is of countable type. From Proposition 3.3.11 we obtain that $\mathrm{Ker}\,p \subset \mathrm{Ker}\,q$.

By Proposition 2.1.27, the map $\varphi : (A/\mathrm{Ker}\, p, \overline{p}) \to (A/\mathrm{Ker}\, q, \overline{q})$ defined by $\varphi(x + \mathrm{Ker}\, p) = x + \mathrm{Ker}\, q \quad (x \in A)$ is a surjective homomorphism. From the fact that q is p-continuous it follows that φ is continuous. By using Proposition 4.2.8 we obtain that $(A/\mathrm{Ker}\, q, \overline{q})$ is of countable type. □

4.2.13 Lemma *Let (A, τ) be a locally convex B_K-module and let p be a seminorm of countable type on A. Let B be a submodule of A. Then $p|B$ is a seminorm of countable type on B.*

Proof: It is not hard to verify that $\varphi : (B/\mathrm{Ker}\, p|B, \overline{p|B}) \to (A/\mathrm{Ker}\, p, \overline{p})$ defined by $\varphi(x + \mathrm{Ker}\, p|B) = x + \mathrm{Ker}\, p \quad (x \in B)$ is a homeomorphism from $(B/\mathrm{Ker}\, p|B, \overline{p|B})$ in $(A/\mathrm{Ker}\, p, \overline{p})$. From the previous proposition we obtain that $(A/\mathrm{Ker}\, p, \overline{p})$ is of countable type and, by Theorem 4.2.7, $(B/\mathrm{Ker}\, p|B, \overline{p|B})$ is of countable type. Again by using the previous proposition we obtain that $p|B$ is of countable type. □

4.2.14 Theorem *Let (A, τ) be a locally convex B_K-module of countable type. Let B be a submodule on A. Then also $(B, \tau|B)$ is of countable type.*

Proof: Let p be a continuous seminorm on B. There exists a bounded seminorm q on B such that $p \sim q$. By Proposition 3.4.20 there exists a continuous seminorm r on A such that $r|B = q$. From Proposition 4.2.12 we obtain that r is of countable type and hence, by Lemma 4.2.13, $q = r|B$ is of countable type. Thus, $(B/\mathrm{Ker}\, q, \overline{q})$ is of countable type. Since $\mathrm{Ker}\, p = \mathrm{Ker}\, q$ and the p-topology equals the q-topology we obtain that the map $1_{B/\ker p} : (B/\mathrm{Ker}\, p, \overline{p}) \to (B/\mathrm{Ker}\, q, \overline{q})$ is a homeomorphism and hence, by Proposition 4.2.8, $(B/\mathrm{Ker}\, p, \overline{p})$ is of countable type. □

4.2.15 Proposition *Let (A, τ) be a locally convex B_K-module. Then:*
A is of countable type $\iff$ For every zero neighbourhood V in A there exists a countable subset X of A such that $A \subset V + \mathrm{co}\, X$.

Proof: $\Rightarrow$) Let V be a zero neighbourhood of A. There exists a continuous seminorm p on A such that $\{x \in A \mid p(x) < 1\} \subset V$. Now $(A/\mathrm{Ker}\, p, \overline{p})$ is of countable type and hence there exists a countable subset Y of $A/\mathrm{Ker}\, p$ such that $\mathrm{co}\, Y$ is dense in $(A/\mathrm{Ker}\, p, \overline{p})$. Let X be a countable subset of A such that $\pi(X) = Y$. We prove that $A = V + \mathrm{co}\, X$. In fact, let $z \in A$. There exists a $\hat{y} \in \mathrm{co}\, Y$ such that $\overline{p}(\pi(z) - \hat{y}) < 1$. Let $y \in \mathrm{co}\, X$ such that $\pi(y) = \hat{y}$. Then $p(z - y) = \overline{p}(\pi(z) - \hat{y}) < 1$ and hence $z - y \in V$. This means that $z \in y + V \subset V + \mathrm{co}\, X$.
$\Leftarrow$) Let p be a continuous seminorm on A. We prove that $(A/\mathrm{Ker}\, p, \overline{p})$ is of countable type. To this end, let $U_n = \{x \in A \mid p(x) < \frac{1}{n}\}$ for each $n \geq 1$. There exist countable subsets $X_1, X_2, X_3, \ldots$ of A such that $A = U_n + \mathrm{co}\, X_n \quad (n \geq 1)$. Let $\pi : A \to A/\mathrm{Ker}\, p$ be the quotient map. Let $Y = \pi(\bigcup_{n \geq 1} X_n)$. We prove that $\mathrm{co}\, Y$ is dense in $(A/\mathrm{Ker}\, p, \overline{p})$. In fact, let $z \in A/\mathrm{Ker}\, p$. Let $\varepsilon > 0$. Let $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. Let $x \in A$ such that $\pi(x) = z$. There exists a $y \in \mathrm{co}\, X_n$ such that $x - y \in U_n$. Then $\pi(y) \in \mathrm{co}\, Y$ and $\overline{p}(z - \pi(y)) = p(x - y) < \frac{1}{n} < \varepsilon$.
We see that $(A/\mathrm{Ker}\, p, \overline{p})$ is of countable type. □

4.2.16 Proposition *A product of locally convex B_K-modules of countable type is of countable type.*

Proof: Let $\{(A_\iota, \tau_\iota)\}_{\iota\in I}$ be a collection of locally convex B_K-modules of countable type. Let $A = \prod_{\iota\in I} A_\iota$. Let V be an open submodule of A. We prove that there exists a countable subset Y of A such that $A \subset V + \operatorname{co} Y$. To this end observe that there exist $n \in \mathbb{N}$, $i_1, \ldots, i_n \in I$ and open submodules $U_1 \subset A_{\iota_1}, \ldots, U_n \subset A_{\iota_n}$ such that $\bigcap_{\iota=1}^{n} P_{\iota_j}^{-1}(U_j) \subset V$. For each $j \in \{1, \ldots, n\}$ there exists a countable subset X_j of A_{ι_j} such that $A_{\iota_j} \subset U_j + \operatorname{co} X_j$. For every $j \in \{1, \ldots, n\}$ let the subset Y_j of A be defined by

$$Y_j = \{x \in A \mid x(i_j) \in X_j \text{ and } x(k) = 0 \text{ if } k \neq i_j\}.$$

Let $Y = \bigcup_{j=1}^{n} V_j$. Then it is not hard to verify that $A \subset V + \operatorname{co} Y$.
By using Proposition 4.2.15 we obtain that A is of countable type. □

4.3 Bounded Sets

The following notion of boundedness is only suitable for Hausdorff locally convex B_K-modules over non-trivially valued K. In this section we therefore assume all locally convex B_K-modules to be Hausdorff.

4.3.1 Definition Let the valuation on K be non-trivial. Let (A, τ) be a locally convex B_K-module. A subset X of A is called *bounded* if for every zero neighbourhood U there exists a $\lambda \in B_K \setminus \{0\}$ such that $\lambda X \subset U$.

4.3.2 Remark If $|K|$ is trivial, then the above definition is not very useful, since with this definition a subset X of a locally convex B_K-module (A, τ) is bounded iff $X \subset \bigcap\{U \mid U \text{ open submodule of } A\}$ and hence:
$X \subset A$ is bounded $\iff$ $p(x) = 0 \quad (x \in X)$ for every τ-continuous seminorm p on A.
Thus, $\{0\}$ is the only bounded subset of A.

In the case that the valuation on K is trivial we make the following definition of a bounded subset.

4.3.3 Definition Let the valuation on K be trivial. Let (A, τ) be a locally convex B_K-module. A subset X of A is called *bounded* if every continuous seminorm on X is bounded.

4.3.4 Remark From K-vector space theory we know the following.

Let the valuation on K be non-trivial. Let (E, τ) be a locally convex K-vector space. Let X be a subset of E. Then: X is bounded $\iff$ every continuous seminorm on E is bounded on X

If $|K|$ is non-trivial then boundedness of a subset X of a locally convex B_K-module A does not imply that every continuous seminorm on X is bounded.

For example, let the valuation on K be dense. Let $(A, \tau) = (B_K^-, |\ |)$. Then B_K^- is bounded. Let ν on B_K^- be defined by $\nu(\lambda) = \frac{|\lambda|}{1-|\lambda|}$ $(\lambda \in B_K^-)$. By Proposition 3.3.15 ν is a norm on B_K^- and $\nu \sim |\ |$. Hence, ν is continuous. But ν is not bounded on B_K^-.

Let $|K|$ be non-trivial. Of course, if every continuous seminorm on a subset X of a locally convex B_K-module (A, τ) is bounded then X is bounded. But there is a lot more to say, also in the case that $|K|$ is trivial. See Theorem 5.2.21.

4.3.5 Theorem *Let the valuation on K be non-trivial. Let (A, τ) be a locally convex B_K-module and $X \subset A$. Then:*
X is bounded $\iff$ $\lim_{\lambda \to 0} p(\lambda x) = 0$ uniformly on X, for every continuous seminorm p on A.

Proof: $\Rightarrow$) Let p be a continuous seminorm on A. Let $\varepsilon > 0$. Then the submodule $U := \{x \in A \mid p(x) < \varepsilon\}$ is open and hence there exists a $\lambda \in B_K \setminus \{0\}$ such that $\lambda X \subset U$. Then $p(\mu x) < \varepsilon \quad (x \in X)$ for every $\mu \in B_K$ with $|\mu| \leq |\lambda|$. We obtain $\lim_{\lambda \to 0} p(\lambda x) = 0$ uniformly on X.
$\Leftarrow$) Let U be a zero neighbourhood. There exists an open submodule V of A such that $V \subset U$. Then p_V is a continuous seminorm and hence $\lim_{\lambda \to 0} p_V(\lambda x) = 0$ uniformly on X. Thus there exists a $\lambda \in B_K \setminus \{0\}$ such that $p_V(\lambda x) < 1 \quad (x \in X)$. Then $\lambda X \subset \{x \in A \mid p_V(x) < 1\} = V \subset U$. □

4.3.6 Proposition *Let (A, τ) be a locally convex B_K-module and let B be a submodule of A. Let $X \subset B$. Then:*
X is bounded in (A, τ) $\iff$ X is bounded in $(B, \tau|B)$.

Proof: Suppose that the valuation on K is trivial. Then A is a K-vector space and B is a subspace of A.
$\Rightarrow$) Let p be a $\tau|B$-continuous seminorm on B. Then p can be extended to a seminorm q on A. Now q is bounded on X and hence so is p.
$\Leftarrow$) Let p be a τ-continuous seminorm on A. Then $p|B$ is $\tau|B$-continuous and hence $p|B$ is bounded on X. As $p = p|B$ on X, also p is bounded on X.
If the valuation on K is non-trivial the proof is easy by using Definition 4.3.1.
□

4.3.7 Examples Let (A, τ) be a locally convex B_K-module such that there exists a $\lambda \in B_K \setminus \{0\}$ with $\lambda A = \{0\}$. Then A is bounded.
If (A, τ) is a locally convex B_K-module then the set $\{x \in A \mid \lambda x = 0\}$ is bounded in A for every $\lambda \in B_K \setminus \{0\}$.

4.3.8 Proposition *Let (A, τ) be a locally convex B_K-module. Then we have the following.*

1. *Finite subsets of A are bounded.*

2. *$X \subset A$ bounded $\Rightarrow$ $\operatorname{co} X$ bounded.*

3. *$X \subset A$ bounded $\Rightarrow$ $\overline{X}$ bounded.*

4. $X \subset Y \subset A$, Y *bounded* $\Rightarrow X$ *bounded.*

5. $X \subset A$ *bounded* $\iff$ *Every countable subset of* X *is bounded.*

6. $X \subset A$ *bounded,* $\lambda \in K \Rightarrow \lambda X$ *bounded.*

Proof: It is not hard to see that the assertions are true if $|K|$ is trivial, so assume K is non-trivial.
1. Let X be a finite subset of A. Let U be an zero neighbourhood in A. Then U is absorbing and as X is finite there exists a $\lambda \in B_K \setminus \{0\}$ such that $\lambda X \subset U$.
2. Let U be a zero neighbourhood in A. There exists an open submodule V of A with $V \subset U$. There exists a $\lambda \in B_K \setminus \{0\}$ such that $\lambda X \subset V$. Then $\lambda \operatorname{co} X = \operatorname{co} \lambda X \subset \operatorname{co} V = V \subset U$.
3. Let U be a zero neighbourhood. Then there exists an open submodule V such that $V \subset U$. Then V is also closed. There exists a $\lambda \in B_K$ such that $\lambda X \subset V$. Then $\lambda \overline{X} \subset \overline{\lambda X} \subset \overline{V} = V \subset U$.
4. trivial
5. $\Rightarrow$ follows from 4.
$\Leftarrow$) Suppose X is not bounded. Then there exists a zero neighbourhood U of A such that $\lambda X \not\subset U$ for every $\lambda \in B_K \setminus \{0\}$. Let $\lambda_1, \lambda_2, \lambda_3, \ldots \in B_K$ such that $|\lambda_1| > |\lambda_2| > |\lambda_3| > \ldots$ and $\lim_{n\to\infty} |\lambda_n| = 0$. Then for every $n \geq 1$ there exists an $x_n \in X$ such that $\lambda_n x_n \notin U$. This is in contradiction with the boundedness of $\{x_1, x_2, x_3, \ldots\}$.
6. Let U be a zero neighbourhood. Let $\mu \in B_K \setminus \{0\}$ such that $\mu X \subset U$. We may suppose that $|\mu| \leq |\lambda|$. Then $|\mu\lambda^{-1}| \leq 1$ and hence, by Proposition 2.1.18, $(\mu\lambda^{-1})(\lambda X) \subset (\mu\lambda^{-1}\lambda)X = \mu X \subset U$. □

4.3.9 Proposition *Let* (A, τ) *and* (B, σ) *be locally convex* B_K*-modules and let* $\varphi : A \to B$ *be a continuous homomorphism. Let* X *be a bounded subset of* A. *Then* $\varphi(X)$ *is a bounded subset of* B.

Proof: Suppose $|K|$ is trivial. Let p be a continuous seminorm on B. Then $p \circ \varphi$ is a continuous seminorm on A. Hence, $p \circ \varphi$ is bounded on X. This implies that p is bounded on $\varphi(X)$. Thus, $\varphi(X)$ is bounded.
Suppose $|K|$ is non-trivial. Let U be a zero neighbourhood in B. Then $\varphi^{-1}(U)$ is a zero neighbourhood in A. Hence, there exists a $\lambda \in B_K \setminus \{0\}$ such that $\lambda X \subset \varphi^{-1}(U)$. Then $\lambda\varphi(X) = \varphi(\lambda X) \subset \varphi(\varphi^{-1}(U)) \subset U$. □

4.3.10 Proposition *Let* I *be an index set and for every* $\iota \in I$ *let* (A_ι, τ_ι) *be a locally convex* B_K*-module. Let* $A = \prod_{\iota \in I} A_\iota$ *and let* τ *be the product topology on* A. *For every* $\iota \in I$ *let* X_ι *be a bounded subset of* A_ι *and let* $X = \prod_{\iota \in I} X_\iota$. *Then* X *is a bounded subset of* (A, τ).

Proof: Let U be a zero neighbourhood of A. Then there exist $n \in \mathbb{N}$, $\iota_1, \ldots, \iota_n \in I$ and open submodules $U_1 \subset A_{\iota_1}, \ldots, U_n \subset A_{\iota_n}$ such that $\bigcap_{j=1}^n P_{\iota_j}^{-1}(U_j) \subset U$. (Here $P_{\iota_j} : A \to A_{\iota_j}$ is the projection map ($j = 1, \ldots, n$)). There exist $\lambda_1, \ldots, \lambda_n \in B_K \setminus \{0\}$ such that $\lambda_j X_{\iota_j} \subset U_j$ for all $j \in \{1, \ldots, n\}$. Let $\mu \in B_K \setminus \{0\}$ such that $|\mu| \leq \min_{1 \leq j \leq n} |\lambda_j|$. Then $P_{\iota_j}(\mu X) = \mu P_{\iota_j}(X) = \mu X_{\iota_j} \subset U_j$ for all $j \in \{1, \ldots, n\}$ and hence $\mu X \subset \bigcap_{j=1}^n P_{\iota_j}^{-1}(U_j) \subset U$. □

4.3.11 Proposition *Let $n \in \mathbb{N}$. If $|K|$ is trivial then K^n is bounded (with respect to the discrete topology). If $|K|$ is non-trivial, then for every $X \subset K^n$: X is bounded (with respect to the unique locally convex Hausdorff topology on K^n, see Proposition 4.1.2) $\Longleftrightarrow$ X is subset of a finitely generated B_K-module.*

Proof: Suppose $|K|$ is trivial. Let $e_1, \ldots, e_n$ be the canonical base for K^n. Then $p \leq \max(p(e_1), \ldots, p(e_n))$ for every seminorm p on A. Hence K^n is bounded with respect to the discrete topology.
Suppose $|K|$ is non-trivial.
$\Rightarrow$) Let $e_1, \ldots, e_n$ be the canonical base for K^n. Let $\| \ \|$ be the max norm with respect to $e_1, \ldots, e_n$. Let $U = \{x \in K^n \mid \|x\| < 1\}$. Then U is open in K^n. Let $\lambda \in B_K \setminus \{0\}$ such that $\lambda X \subset U$.
Let $x \in X$. There exist $\mu_1, \ldots, \mu_n \in K$ such that $x = \mu_1 e_1 + \ldots + \mu_n e_n$. Then $\lambda x = \lambda \mu_1 e_1 = \ldots + \lambda \mu_n e_n$ and $\|\lambda x\| < 1$. Thus, $|\lambda \mu_i| < 1$ and hence $|\mu_i| < |\lambda^{-1}|$ for all $i \in \{1, \ldots, n\}$.
We obtain that $X \subset \operatorname{co}\{\lambda^{-1} e_1, \ldots, \lambda^{-1} e_n\}$.
$\Leftarrow$) Let Y be a finite subset of K^n such that $X \subset \operatorname{co} Y$. With 1. and 2. of Proposition 4.3.8 we obtain that $\operatorname{co} Y$ is bounded and with 4. of the same proposition we obtain that X is bounded. □

Now we study boundedness in arbitrary modules in $\mathcal{F}_K$.

4.3.12 Proposition *Let $A \in \mathcal{F}_K$. Then the following assertions are equivalent.*

(i) *A is a homomorphic image of a bounded absolutely convex subset of K^n for some $n \in \mathbb{N}$ (i.e. $A \in \mathcal{B}_K$).*

(ii) *A is a subset of a finitely generated B_K-module.*

(iii) *A is bounded with respect to every locally convex Hausdorff topology on A.*

(iv) *A is bounded with respect to some locally convex Hausdorff topology on A.*

Proof: If $|K|$ is trivial then the assertions (i)-(iv) are all true and there remains nothing to prove. Hence, suppose $|K|$ is non-trivial.
(i) $\Rightarrow$ (ii) is Proposition 2.2.40.
(ii) $\Rightarrow$ (iii): Let C be a finitely generated B_K-module such that $A \subset C$. There exists a finitely generated absolutely convex subset D of some K^n and a surjective homomorphism $\varphi : D \to C$. Let $B = \varphi^{-1}(A)$. By Proposition 4.3.11 B is bounded. Then $\varphi | B : (B, \tau_B) \to (A, \tau_A)$ is continuous (see Proposition 4.1.7) and hence (A, τ_A) is bounded. Let τ be a locally convex Hausdorff topology on A. Then τ is weaker than τ_A and hence (A, τ) is also bounded.
(iii) $\Rightarrow$ (iv): τ_A is a locally convex Hausdorff topology on A and from (iii) it follows that (A, τ_A) is bounded.
(iv) $\to$ (i): Let τ be a locally convex Hausdorff topology on A such that (A, τ) is bounded. Let $n = \operatorname{rank} A$. Let B be an absolutely convex subset of K^n and let $\varphi : B \to A$ be a surjective homomorphism. Let $\nu \in K$, $|\nu| > 1$. By using Proposition 2.2.37 we obtain that there exists an elementary subset X of K^n

such that $B \subset X \subset \nu B$. Let $m \in \mathbb{N}$, let $x_1, \dots, x_n \in K^n$ and let $C_1, \dots, C_m$ be absolutely convex subsets of K such that $X = C_1x_1 + \cdots + C_mx_m$. We prove that each C_ι is bounded.
Let $i \in \{1, \dots, m\}$. According to Proposition 2.2.28 we obtain that there exists a $\lambda \in K \setminus \{0\}$ such that $\lambda x_\iota \in B$ and $\varphi(\lambda x_\iota) \neq 0$. As (A, τ) is Hausdorff there exists a τ-open submodule U such that $\varphi(\lambda x_\iota) \notin U$. Since (A, τ) is bounded there exists a $\mu \in B_K \setminus \{0\}$ such that $\mu A \subset U$. Let $\rho \in K$ such that $|\rho| > |\lambda\mu^{-1}\nu|$ and suppose that $\rho \in C_\iota$. Then $\nu^{-1}\rho x_\iota \in B$. Then $|\lambda\nu\mu^{-1}\rho^{-1}| < 1$ and hence $\lambda\nu\mu^{-1}\rho^{-1} \in B_K$. Now

$$\varphi(\lambda x_\iota) = \varphi((\lambda\nu\mu^{-1}\rho^{-1})\mu\nu^{-1}\rho x_\iota) = (\lambda\nu\mu^{-1}\rho^{-1})\varphi(\mu(\nu^{-1}\rho x_\iota))$$

and $(\lambda\nu\mu^{-1}\rho^{-1})\varphi(\mu(\nu^{-1}\rho x_\iota)) = (\lambda\nu\mu^{-1}\rho^{-1})(\mu\varphi(\nu^{-1}\rho x_\iota))$. Now

$$(\lambda\nu\mu^{-1}\rho^{-1})(\mu\varphi(\nu^{-1}\rho x_\iota)) \in (\lambda\nu\mu^{-1}\rho^{-1})U \subset U$$

and thus $\varphi(\lambda x_\iota) \in U$, a contradiction. Hence, $C_\iota \subset \{\eta \in K \mid |\eta| \leq |\lambda\mu^{-1}\nu|\}$, which implies that C_ι is bounded. Then also $C_1x_1 + \cdots + C_mx_m$ is bounded and hence so is B. □

4.3.13 Corollary *Let $A \in \mathcal{F}_K$. Let τ be a locally convex Hausdorff topology on A and let $X \subset A$. Then:*
X is bounded in (A, τ) $\iff$ X is bounded in (A, τ_A).

An absolutely convex subset A of a K-vector space E that does not contain linear spaces except $\{0\}$ is bounded with respect to some Hausdorff locally convex topology on $[A]$, namely the topology induced by the Minkowsky function on $[A]$. To find out generalizations to B_K-modules (see Theorem 4.3.19) we introduce the notion of a divisible element.

4.3.14 Definition Let A be a B_K-module. An $x \in A$ with $x \neq 0$ is called a *divisible element* if for every $\lambda \in B_K \setminus \{0\}$ there exists a $y \in A$ such that $\lambda y = x$.

A B_K-module which has divisible elements need not to contain a non-trivial divisible submodule as we see in the following example.

4.3.15 Example Let the valuation on K be non-trivial. Let $e_0, e_1, e_2, \dots$ be the canonical unit vectors of $K^{(\mathbb{N})}$. Let $\lambda \in K$, $|\lambda| > 1$. Let

$$B = \text{co}\,\{e_0, \lambda e_1, \lambda^2 e_2, \dots\}$$

and let

$$C = \text{co}\,\{e_0 - e_1, e_0 - e_2, e_0 - e_3, \dots\}.$$

We first observe that $C \subset B$. Let $A := B/C$. We shall prove that $e_0 + C$ is a divisible element of A, but that A contains no divisible submodules but $\{0\}$. The proof is divided into several steps.
1. *$e_0 + C$ is a divisible element*
Proof: Let $\mu \in B_K$, $\mu \neq 0$. Let $n \in \mathbb{N}$ such that $|\lambda|^{-n} \leq |\mu| < |\lambda|^{-n+1}$. Then

$\lambda^{-n}\mu^{-1} \in B_K$. Now $\mu^{-1}e_n + C = (\lambda^{-n}\mu^{-1})\lambda^n e_n + C \in A$ and $\mu(\mu^{-1}e_n + C) = e_n + C = e_0 + C$. □

2. *Let $\mu_0, \dots, \mu_n \in B_K$. Then $\mu_0 e_0 + \cdots + \mu_n e_n + C = (\sum_{i=0}^n \mu_i)e_0 + C$. Furthermore, if $\mu \in B_K$ such that $\mu e_0 \in C$ then $\mu = 0$.*
Proof: $\mu_0 e_0 + \cdots + \mu_n e_n + C = (\sum_{i=0}^n \mu_i)e_0 + \sum_{j=1}^n(-\mu_j)(e_0 - e_j) + C = (\sum_{i=0}^n \mu_i)e_0 + C$.
Furthermore, let $\mu \in B_K$ such that $\mu e_0 \in C$. Then there exist $n \in \mathbb{N}$ and $\nu_1, \dots \nu_n \in B_K$ such that $\mu e_0 = \nu_1(e_0 - e_1) + \dots + \nu_n(e_0 - e_n)$. That is to say $(\mu - \sum_{i=1}^n \nu_i)e_0 + \nu_1 e_1 + \cdots + \nu_n e_n = 0$ and hence $\mu - \sum_{i=1}^n \nu_i = 0$ and $\nu_1 = \dots = \nu_n = 0$. This implies $\mu = 0$. □

3. *Let $\mu_1, \dots, \mu_n \in B_K$ and let $x = \mu_0 e_0 + \mu_1\lambda e_1 + \dots + \mu_n\lambda^n e_n \in B$. Then: $x \in C \iff \mu_0, \mu_1\lambda, \dots, \mu_n\lambda^n \in B_K$ and $\mu_0 + \mu_1\lambda + \cdots + \mu_n\lambda^n = 0$.*
Proof: $\Leftarrow$) By 2., $x + C = (\sum_{i=0}^n \mu_i\lambda^i)e_0 + C = 0e_0 + C = C$ and hence $x \in C$.
$\Rightarrow$) There exist $m \geq n$ and $\nu_1, \dots, \nu_m \in B_K$ such that $x = \sum_{j=1}^n \nu_j(e_0 - e_j)$. Then

$$(\mu_0 - \sum_{i=1}^m \nu_i)e_0 + \sum_{j=1}^n(\mu_j\lambda^j + \nu_j)e_j + \sum_{j=n+1}^n \nu_j e_j = \sum_{j=0}^n \mu_j\lambda^j e_j - \sum_{j=1}^m \nu_j(e_j - e_0) = x - x = 0.$$

This implies that $\mu_0 - \sum_{i=1}^m \nu_i = 0$, $\mu_1\lambda + \nu_1 = \dots = \mu_n\lambda^n + \nu_n = 0$ and $\nu_{n+1} = \dots = \nu_m = 0$.
Then, in particular, $\mu_0 = \sum_{i=1}^n \nu_i \in B_K$ and $\mu_i\lambda^i = -\nu_i \in B_K$ for $i = 1, \dots, n$. Furthermore, $\mu_0 + \mu_1\lambda + \cdots + \mu_n\lambda^n = \mu_0 - \sum_{i=1}^n \nu_i = 0$. □

4. *Let $y \in A$. Let $n \in \mathbb{N}$ and $\nu_0, \dots, \nu_n \in B_K$ such that $y = \sum_{j=0}^n \nu_j\lambda^j e_j + C$. Suppose there exists an $i \in \{1, \dots, n\}$ such that $|\nu_i\lambda^i| > 1$. Then y is not a divisible element of A.*
Proof: Let $k \in \{1, \dots, n\}$ such that $|\nu_k\lambda^k| > 1$. Suppose there exists a $z \in A$ such that $\lambda^{-k}z = y$.
Let $m \geq n$ and let $\rho_0, \dots, \rho_m \in B_K$ be such that $z = \sum_{j=1}^m \rho_j e_j + C$. Then

$$\sum_{j=1}^m \rho_j\lambda^{j-k}e_j + C = \lambda^{-k}z = y = \sum_{j=0}^n \nu_j\lambda^j e_j + C$$

and hence $\sum_{j=0}^n(\rho_j\lambda^{j-k} - \nu_j\lambda^j)e_j + \sum_{j=n+1}^m \rho_j\lambda^{j-k}e_j \in C$. From 3. it follows that $\rho_k - \nu_k\lambda^k \in B_K$. But $\rho_k \in B_K$ and $\nu_k\lambda^k \notin B_K$, a contradiction.
We obtain that y is not divisible. □

5. *Let D be a divisible submodule of A. Then $e_0 + C \notin D$.*
Proof: Suppose $e_0 + C \in D$.
Let $\mu \in B_K^-$ and $y \in D$ such that $\mu y = e_0 + C$.
Let $n \in \mathbb{N}$ and $\nu_0, \dots, \nu_n \in B_K$ such that $y = \sum_{j=0}^n \nu_i\lambda^i e_i + C$. Suppose

that $\nu_i\lambda^i \in B_K$ for all $i \geq 1$. Then $e_0 + C = \mu y = \sum_{j=0}^{n} \mu\nu_i\lambda^i e_i + C = (\mu\sum_{i=0}^{n} \nu_i\lambda^i)e_0 + C$ and hence $((1-\mu\nu_0) - \mu\sum_{i=0}^{n} \nu_i\lambda^i)e_0 \in C$. This implies that $(1-\mu\nu_0) - \mu\sum_{i=0}^{n} \nu_i\lambda^i = 0$. But $|1-\mu\nu_0| = 1$ and $|\mu\sum_{i=0}^{n} \nu_i\lambda^i| \leq |\mu| < 1$, a contradiction.
Thus there exists an $i \in \{1, \ldots, n\}$ with $|\nu_i\lambda^i| > 1$ and this implies that y is not divisible, again a contradiction. Hence, $e_0 + C \notin D$. □

6. *Let D be a divisible submodule of A. Then $D = \{0\}$.*
Proof: Let $y \in D$. There exist $n \in \mathbb{N}$ and $\mu_0, \ldots, \mu_n \in B_K$ such that $y = \sum_{j=0}^{n} \mu_j\lambda^j e_j + C$. Now y is divisible and hence $\mu_i\lambda^i \in B_K$ for all $i \in \{0, \ldots, n\}$. Then $y = (\sum_{i=0}^{n} \mu_i\lambda^i)e_0 + C$. Let $\nu = \sum_{i=0}^{n} \mu_i\lambda^i$.
Suppose $\nu \neq 0$. Then there exists a $z \in D$ such that $\nu z = y$. Let $m \in \mathbb{N}$ and $\rho_0, \ldots, \rho_m \in B_K$ such that $z = \sum_{j=0}^{m} \rho_j\lambda^j e_j + C$. As z is divisible we obtain that $\rho_i\lambda^i \in B_K$ for all $i \in \{0, \ldots, m\}$.
Now $\nu e_0 + C = y = \nu z = \sum_{j=0}^{m} \nu\rho_j\lambda^j e_j + C = (\sum_{i=1}^{m} \nu\rho_i\lambda^i)e_0 + C$ and hence $(\nu - \sum_{i=0}^{m} \nu\rho_i\lambda^i)e_0 \in C$. Then 2. implies that $\sum_{i=0}^{m} \rho_i\lambda^i = 1$ and hence $z = e_0 + C$, a contradiction.
Thus, $\nu = 0$ and hence $y = 0$. We obtain that $D = \{0\}$. □

4.3.16 Theorem *Let A be a B_K-module that has no divisible elements. Then A is a homomorphic image of an absolutely convex subset of some K-vector space that contains no linear subspaces apart from $\{0\}$.*

Proof: According to Proposition 2.1.33 there exists an absolutely convex subset B of some vector space E and a surjective homomorphism $\varphi : B \to A$. Let $D = \{x \in B \mid [x] \subset B\}$. Then D is a linear subspace of E. Let F be a linear subspace of E such that $D \oplus F = E$ and let $C = F \cap B$. It it is not hard to verify that $B = D \oplus C$. Now C is an absolutely convex subset of E that contains no linear subspaces but $\{0\}$.
Let $x \in D$. Let $\mu \in B_K \setminus \{0\}$. Then $\mu^{-1}x \in D$ and $\mu\varphi(\mu^{-1}x) = \varphi(x)$. We see that $\varphi(x)$ is a divisible element of A. Hence $\varphi(x) = 0$.
We obtain that $D \subset \operatorname{Ker}\varphi$. From Proposition 2.1.28 it follows that there exists a surjective homomorphism from B/D to A. As $B = D \oplus C$ we obtain that B/D is isomorphic to C. And hence there exists also a surjective homomorphism from C to A. □

4.3.17 Remark If the valuation on K is non-trivial, then the converse of the previous theorem is not true. A B_K-module A that is a homomorphic image of an absolutely convex subset that contains no linear subspaces may have divisible elements. It is even possible that A is a K-vector space !
For example, let

$$c_0 := \{(\lambda_0, \lambda_1, \lambda_2, \ldots) \mid \lambda_i \in K \text{ for all } i \in \mathbb{N} \text{ and } \lim_{n\to\infty} \lambda_n = 0\}.$$

Let

$$B_0 = \{(\lambda_0, \lambda_1, \lambda_2, \ldots) \mid \lambda_i \in B_K \text{ for all } i \in \mathbb{N} \text{ and } \lim_{n\to\infty} \lambda_n = 0\}$$

be the unit ball of c_0 and let

$$B_{00} = \{(\lambda_0, \lambda_1, \lambda_2, \ldots) \mid \lambda_i \in B_K \text{ for all } i \in \mathbb{N} \text{ and } \lambda_i = 0 \text{ for large } i\}.$$

Let $A = B_0/B_{00}$.
B_0 is an absolutely convex subset of c_0 that contains no non-trivial linear subspaces and the quotient map $\pi : B_0 \to A$ is a surjective homomorphism.
Now A is divisible. For let $x \in A$ and $\mu \in B_K \setminus \{0\}$. Let $\lambda_0, \lambda_1, \lambda_2, \ldots \in B_K$ with $\lim_{n\to\infty} \lambda_n = 0$ such that $x = (\lambda_0, \lambda_1, \lambda_2, \ldots) + B_{00}$.
Let $N \in \mathbb{N}$ such that $|\lambda_n| < |\mu|$ for all $n \geq N$. Then

$$x = (\lambda_0, \lambda_1, \lambda_2, \ldots) + B_{00} = (0, \ldots, 0, \lambda_N, \lambda_{N+1}, \ldots) + B_{00}.$$

Now $\mu^{-1}\lambda_n \in B_K$ for all $n \geq N$ and $\lim_{n\to\infty} \mu^{-1}\lambda_n = 0$. Thus

$$(0, \ldots, 0, \mu^{-1}\lambda_N, \mu^{-1}\lambda_{N+1}, \ldots) + B_{00} \in A$$

and

$$\mu(0, \ldots, 0, \mu^{-1}\lambda_N, \mu^{-1}\lambda_{N+1}, \ldots) + B_{00} = (0, \ldots, 0, \lambda_N, \lambda_{N+1}, \ldots) + B_{00} = x.$$

We see that for every $x \in A$ and every $\mu \in B_K \setminus \{0\}$ there exists a $y \in A$ such that $\mu y = x$. Hence, A is divisible.
A is also torsion free. In fact, let $x \in A$ and $\mu \in B_K$ such that $\mu x = 0$. Let $\lambda_0, \lambda_1, \lambda_2, \ldots \in B_K$ with $\lim_{n\to\infty} \lambda_n = 0$ such that $x = (\lambda_0, \lambda_1, \lambda_2, \ldots) + B_{00}$. Then $\mu x = (\mu\lambda_0, \mu\lambda_1, \mu\lambda_2, \ldots) + B_{00}$. As $\mu x = 0$ we obtain that $(\mu\lambda_0, \mu\lambda_1, \mu\lambda_2, \ldots) \in B_{00}$ and hence there exists an $N \in \mathbb{N}$ such that $\mu\lambda_n = 0$ for all $n \geq N$. Then $\mu = 0$ or $\lambda_n = 0$ for every $n \geq N$. In the later case we have that $x = (\lambda_0, \lambda_1, \lambda_2, \ldots) + B_{00} = (\lambda_0, \ldots, \lambda_{N-1}, 0, 0, \ldots) + B_{00} = B_{00} = 0$. We see that if $x \in A$ and $\mu \in B_K$ such that $\mu x = 0$ then $\mu = 0$ or $x = 0$. Thus, A is torsion free.
We obtain that A is a torsion free divisible B_K-module and hence, by Theorem 2.1.14, a K-vector space.

The proof of the following proposition is straightforward.

4.3.18 Proposition *Let A be a B_K-module. Then*

$$\bigcap_{\lambda \in B_K \setminus \{0\}} \lambda A = \{0\} \cup \{x \in A \mid x \text{ is divisible}\}.$$

If A is an absolutely convex set then $\bigcap_{\lambda \in B_K \setminus \{0\}} \lambda A$ is the largest linear subspace contained in A.

4.3.19 Theorem *Let A be a B_K-module. Then the following assertions are equivalent.*

(i) *A has no divisible elements.*

(ii) *There exists an absolutely convex set B which contains no linear subspaces $\neq \{0\}$ and a surjective homomorphism $\varphi : B \to A$ such that* $\operatorname{Ker}\varphi$ *is closed with respect to the q_B-topology on B.*

(iii) *There exists a locally convex Hausdorff topology τ on A such that (A, τ) is bounded.*

Proof: (i) $\Rightarrow$ (ii): From Theorem 4.3.16 we obtain that there exist an absolutely convex set C that contains no linear subspaces $\neq \{0\}$ and a surjective homomorphism $\varphi : C \to A$. We prove that $\operatorname{Ker}\varphi$ is closed with respect to the q_C-topology on C. To this end, let $\overline{\operatorname{Ker}\varphi}$ be the closure of $\operatorname{Ker}\varphi$ with respect to the q_C-topology. Let $x \in \overline{\operatorname{Ker}\varphi}$. Let $\lambda \in B_K \setminus \{0\}$. Then λC is q_C-open and hence $(x + \lambda C) \cap \operatorname{Ker}\varphi \neq \emptyset$. Let $y \in C$ such that $x + \lambda y \in \operatorname{Ker}\varphi$. Then $\varphi(x) = \lambda\varphi(-y)$.
We see that for every $\lambda \in B_K \setminus \{0\}$ there exists a $z \in A$ such that $\lambda z = \varphi(x)$. Thus, $\varphi(x)$ is divisible and hence $\varphi(x) = 0$. That is to say $x \in \operatorname{Ker}\varphi$.
We obtain $\overline{\operatorname{Ker}\varphi} = \operatorname{Ker}\varphi$.
(ii) $\Rightarrow$ (iii): Let B be an absolutely convex set that contains no linear subspaces $\neq \{0\}$ and let $\varphi : B \to A$ a surjective homomorphism such that $\operatorname{Ker}\varphi$ is closed with respect to the q_B-topology. Now q_B is a norm on B for B contains no linear subspaces. Furthermore, B is bounded with respect to q_B since $(\lambda B)_{\lambda \in B_K \setminus \{0\}}$ is a base of zero neighbourhoods for the q_B-topology. Let p be the quotient norm of q_B on $B/\operatorname{Ker}\varphi$. Then $(B/\operatorname{Ker}\varphi, p)$ is a locally convex B_K-module and as the quotient map $\pi : (B, q_B) \to (B/\operatorname{Ker}\varphi, p)$ is continuous we obtain that $(B/\operatorname{Ker}\varphi, p)$ is bounded. As $A \sim B/\operatorname{Ker}\varphi$ we obtain that there exists a locally convex Hausdorff topology τ on A such that (A, τ) is bounded.
(iii) $\Rightarrow$ (i): Let τ be a locally convex Hausdorff topology on A such that (A, τ) is bounded. Let x be a divisible element of A. Let U be an open submodule of A and let $\lambda \in B_K \setminus \{0\}$ such that $\lambda A \subset U$. Let $y \in A$ such that $\lambda y = x$. Then $x = \lambda y \in U$. We obtain that $x \in U$ for every open submodule U of A. As τ is Hausdorff it follows that $x = 0$. □

4.3.20 Theorem *Let A be a B_K-module without divisible elements. Then q_A is a norm on A and the q_A-topology is the strongest Hausdorff locally convex topology τ on A such that (A, τ) is bounded.*

Proof: 1. Let $x \in A$ such that $q_A(x) = 0$. For every $\lambda \in B_K \setminus \{0\}$ we have that $x \in \lambda A$ and hence $x \in \bigcap_{\lambda \in B_K \setminus \{0\}} \lambda A$. From Proposition 4.3.18 it follows that $x = 0$.
2. By Proposition 3.3.33 the collection $(\lambda A)_{\lambda \in B_K \setminus \{0\}}$ is a base of zero neighbourhoods for the q_A-topology which implies that (A, q_A) is bounded.
3. Let τ be a locally convex Hausdorff topology on A such that A is bounded with respect to τ. Let U be a τ-open submodule. There exists a $\lambda \in B_K \setminus \{0\}$ such that $\lambda A \subset U$. As λA is q_A-open it follows that also U is q_A-open. Thus, τ is weaker than the q_A-topology. □

4.3.21 Proposition *Let B be an absolutely convex set that contains no linear subspaces $\neq \{0\}$. Let C be a B_K-submodule of B that is closed with respect to the q_B-topology. Then the quotient norm of q_B on B/C equals $q_{B/C}$.*

Proof: Let p be the quotient norm of q_B on B/C.
case 1. $|K|$ is discrete. Let $x \in B$. Let $\lambda \in K$ such that $|\lambda| = q_{B/C}(x + C)$. Then $x + C \in \lambda(B/C)$, hence there exists a $y \in \lambda B$ such that $x + C = y + C$. Then $p(x + C) = p(y + C) \leq q_B(y) \leq |\lambda| = q_{B/C}(x + C)$.
As $|K|$ is discrete there exists a $c \in C$ such that $p(x + C) = q_B(x + c)$. Let

$\mu \in K$ such that $q_B(x+c) = |\mu|$. Then $x+c \in \mu B$ and hence $x+C \in \mu(B/C)$. Then $q_{B/C}(x+C) \le |\mu| = p(x+C)$.
We obtain that $p = q_{B/C}$.
case 2. $|K|$ is dense. Let $x \in B$. Let $\lambda \in K$ such that $|\lambda| > q_{B/C}(x+C)$. Then $x+C \in \lambda(B/C)$ and hence, as in **case 1.**, $p(x+C) \le |\lambda|$. As $|K|$ is dense we obtain that $p(x+C) \le q_{B/C}(x+C)$.
Let $\lambda \in B_K$ such that $|\lambda| > p(x+C)$. Then there exists a $c \in C$ such that $q_B(x+c) < |\lambda|$. Thus, $x+c \in \lambda B$ and hence, as in **case 1.**, $q_{B/C}(x+C) \le |\lambda|$. As $|K|$ is dense we obtain that $q_{B/C}(x+C) \le p(x+C)$.
We see that $p = q_{B/C}$. □

4.3.22 Remark Let $|K|$ be non-trivial. Let A be a B_K-module that is bounded with respect to every locally convex Hausdorff topology on A. Then not necessarily $A \in \mathcal{B}_K$.
For example, let $\lambda \in B_K$ with $0 < |\lambda| < 1$. Let $A = B_K^{\mathbb{N}}/\lambda B_K^{\mathbb{N}}$. Then $\lambda A = \{0\}$ and hence A is bounded with respect to every locally convex topology on A. But not $A \in \mathcal{F}_K$, which can be seen as follows. Let $e_0, e_1, e_2, \ldots$ be the canonical unit vectors in $B_K^{\mathbb{N}}$. Let $\pi : B_K^{\mathbb{N}} \to A$ be the quotient map. From Proposition 2.2.10 we obtain that $\mathrm{co}\,\{\pi(e_0), \ldots, \pi(e_{n-1})\}$ is n-generated for every $n \in \mathbb{N}$. By using Proposition 2.2.35 we can conclude that A can not be of finite rank.

4.3.23 Theorem *Let A be a B_K-module without divisible elements. Then the following assertions are equivalent.*

(i) *A is bounded with respect to every locally convex Hausdorff topology on A.*

(ii) *A is bounded with respect to the strongest locally convex Hausdorff topology on A.*

(iii) *$\lambda A \in \mathcal{B}_K$ for some $\lambda \in B_K \setminus \{0\}$.*

Proof: (i) ⇒ (ii): trivial.
(ii) ⇒ (iii): Let A_t be the torsion part of A. Then A_t is bounded with respect to the strongest locally convex topology on A_t, τ_{A_t}, since, by Lemma 4.1.11, $\tau_{A_t} = \tau_A | A_t$. As A_t is a torsion module, every submodule of A_t is absorbing and hence τ_{A_t} is discrete. Then $\{0\}$ is open and hence there exists a $\lambda \in B_K \setminus \{0\}$ such that $\lambda A_t = \{0\}$. Let $S_\lambda : A \to A$ be defined by $S_\lambda(x) = \lambda x \quad (x \in A)$. Then $\mathrm{Im}\, S_\lambda = \lambda A$ and $\mathrm{Ker}\, S_\lambda = A_t$. Thus $\lambda A \sim A/A_t$. Now A is bounded with respect to τ_A and hence A/A_t is bounded with respect to the quotient topology of τ_A. By Lemma 4.1.11 the quotient topology on A/A_t equals τ_{A/A_t}. Now A/A_t is torsion free and hence embeddable in a vector space E such that $[A/A_t] = E$ and A/A_t is bounded with respect to any locally convex Hausdorff (module) topology on E. Then E must be finite dimensional and hence, $A/A_t \in \mathcal{F}_K$. From Proposition 4.3.12 we obtain that $A/A_t \in \mathcal{B}_K$ which implies that also $\lambda A \in \mathcal{B}_K$.
(iii) ⇒ (i): Let τ be a locally convex Hausdorff topology on A. Then, by Proposition 4.3.12, λA is bounded with respect to $\tau | \lambda A$ and hence λA is bounded in A with respect to τ. Now $A = \lambda^{-1}(\lambda A)$ and hence, by 6. of Proposition 4.3.8, A is bounded with respect to τ. □

4.4 τ^- and τ^+

In this technical section we will introduce for locally convex B_K-modules (A,τ) over a densely valued K two related locally convex topologies: τ^- and τ^+. These topologies are typical for B_K-modules; their counterparts in K-vector space theory are trivial. That is to say that for every locally convex K-vector space (E,τ) we have $\tau^- = \tau = \tau^+$. They play an important role in the next chapter, where they will be used to study the (non-)openness of surjections between c-compact sets.

In this section we assume that the valuation on K is dense unless clearly stated otherwise.

The Minus Topology

4.4.1 Definition Let (A,τ) be a locally convex B_K-module. A subset V of A is called τ^-*-open* if for every $y \in V$ there exists a τ-open submodule U and a $\lambda \in K$, $|\lambda| > 1$ such that $y + \lambda U \subset V$.

4.4.2 Remark We could also define the minus topology for locally convex B_K-modules (A,τ) for arbitrary valued fields K (so as to include discrete valued K) by calling a subset V of A τ^--open if for every $y \in V$ there exists an $r > 1$ and a τ-open submodule U of A such that $y + \lambda U \subset V$ for every $\lambda \in K$ with $1 \leq |\lambda| \leq r$.
This definition is equivalent with the above one if $|K|$ is dense. If $|K|$ is discrete then $\tau^- = \tau$.

4.4.3 Proposition *Let (A,τ) be a locally convex B_K-module. Then τ^- is a locally convex topology on A.*

Proof: Let C be the collection of all τ^--open submodules of A.
(i) Every member of C is absorbing. In fact, let U be a member of C. Then there exists a τ-open submodule V and a $\lambda \in K$, $|\lambda| > 1$ such that $\lambda V \subset U$. Now λV is absorbing since V is absorbing and hence also U is absorbing.
(ii) For every finite subcollection $\mathcal{F}$ of C there exists a $U \in C$ such that $U \subset \bigcap \mathcal{F}$. In fact, let $\mathcal{F} = \{U_1, \ldots, U_n\}$ be a finite subcollection of C. Let $V_1, \ldots, V_n$ be τ-open submodules and let $\lambda_1, \ldots, \lambda_n \in K$, $|\lambda_1|, \ldots, |\lambda_n| > 1$ such that $\lambda_i V_i \subset U_i$ for all $i \in \{1, \ldots, n\}$. Let $V = \bigcap_{i=1}^n V_i$ and let $\lambda \in K$ be such that $1 < |\lambda| < \min\{|\lambda_1|, \ldots, |\lambda_n|\}$. Then $\lambda V \in C$ and $\lambda V \subset \bigcap \mathcal{F}$.
Now τ^- equals the C-topology and by Proposition 3.1.26 this implies that τ^- is locally convex. □

4.4.4 Proposition *Let (A,τ) and (B,σ) be locally convex B_K-modules. Let $\varphi : (A,\tau) \to (B,\sigma)$ be a continuous homomorphism. Then the map $\varphi : (A,\tau^-) \to (B,\sigma^-)$ is also continuous.*

Proof: Let V be a σ^--open submodule of B. There exists a $\lambda \in K$, $|\lambda| > 1$ and a σ-open submodule U of B such that $\lambda U \subset V$. Then, by 2. of Proposition 2.1.20, $\lambda\varphi^{-1}(U) = \varphi^{-1}(\lambda U) \subset \varphi^{-1}(V)$. As $\varphi^{-1}(U)$ is τ-open we conclude

that $\varphi^{-1}(V)$ is τ^--open.
Hence, $\varphi : (A, \tau^-) \to (B, \sigma^-)$ is continuous. □

4.4.5 Proposition *Let (A, τ) be a locally convex B_K-module. Let B be a submodule of A. Then $\tau^-|B = (\tau|B)^-$.*

Proof: From the continuity of the inclusion map $(B, \tau|B) \to (A, \tau)$ and the previous proposition it follows that $\tau^-|B \leq (\tau|B)^-$.
Now let U be a $(\tau|B)^-$ open submodule of B. Then there exist a $\lambda \in K$ with $|\lambda| > 1$ and a $\tau|B$-open submodule V of B such that $\lambda V \subset U$. (Here $\lambda V = \{x \in B \mid \lambda^{-1}x \in V\}$.) By proposition 3.1.30 there exists a τ-open submodule W of A such that $V = W \cap B$. Then λW is a τ^--open submodule of A and thus $\lambda W \cap B$ is a $\tau^-|B$-open submodule of B. Furthermore, $\lambda W \cap B = \lambda V \subset U$ and hence also U is $\tau^-|B$-open.
We see that also $(\tau|B)^- \leq \tau^-|B$. □

4.4.6 Proposition *Let (A, τ) be a locally convex B_K-module. Then:*

1. $\tau^- \leq \tau$.

2. $\tau^{--} = \tau^-$.

Proof: **1.** Let V be a τ^--open submodule of A. Then there exists a $\lambda \in K$, $|\lambda| > 1$ and a τ-open submodule U such that $\lambda U \subset V$. Then $U \subset \lambda U \subset V$ and hence V is also τ-open.
2. Let V be a τ^--open submodule. Let $\lambda \in K$ with $|\lambda| > 1$ and let U be an open submodule of A such that $\lambda U \subset V$. Let $\mu \in K$, $|\mu| > 1$ such that $|\mu|^2 < |\lambda|$.
Then μU is τ^--open and $\mu(\mu U) = \mu^2 U \subset \lambda U \subset V$.
We see that every τ^--open submodule is also τ^{--}-open. Together with 1. we obtain $\tau^{--} = \tau^-$. □

4.4.7 Proposition *Let (A, τ) be a locally convex B_K-module. Let p be a seminorm on A and let $\lambda \in B_K$. Then $p_\lambda : A \to [0, \infty)$ defined by*

$$p_\lambda(x) = p(\lambda x) \qquad (x \in A)$$

is a seminorm on A.

We leave the proof to the reader.

4.4.8 Theorem *Let (A, τ) be a locally convex B_K-module. Let $\mathcal{P}$ be a generating collection of seminorms for τ. Then τ^- is generated by the seminorms $(p_\mu)_{\mu \in B_K^-,\, p \in \mathcal{P}}$.*

Proof: **1.** Let $p \in \mathcal{P}$ and let $\mu \in B_K^-$. Let $\varepsilon > 0$. Then

$$\{x \in A \mid p_\mu(x) < \varepsilon\} = \{x \in A \mid p(\mu x) < \varepsilon\} = \mu^{-1}\{y \in A \mid p(y) < \varepsilon\}.$$

The latter set is τ^--open and hence so is $\{x \in A \mid p_\mu(x) < \varepsilon\}$.
From Proposition 3.3.7 we obtain that p_μ is τ^--continuous.

2. Let V be a τ^--open submodule. There exist a $\lambda \in K$ with $|\lambda| > 1$ and a τ-open submodule U such that $\lambda U \subset V$.
There exist an $n \in \mathbb{N}$, $p_1, \ldots, p_n \in \mathcal{P}$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that $\bigcap_{i=1}^n \{x \in A \mid p_i(x) < \varepsilon_i\} \subset U$. Now

$$\bigcap_{i=1}^{n} \{x \in A \mid (p_i)_{\lambda^{-1}}(x) < \varepsilon_i\} = \bigcap_{i=1}^{n} \{x \in A \mid p_i(\lambda^{-1}x) < \varepsilon_i\} =$$
$$\lambda \bigcap_{i=1}^{n} \{x \in A \mid p_i(x) < \varepsilon_i\} \subset \lambda U \subset V.$$

We see that $(p_\mu)_{\mu \in B_K^-,\, p \in \mathcal{P}}$ is a collection of τ^--continuous seminorms on A that generates τ^-. □

4.4.9 Remark In general, the collection

$$(p_\mu)_{\mu \in B_K^-,\ p\ \tau\text{-continuous seminorm on } A}$$

is not the collection of all τ^--continuous seminorms on A. It is even not always a $*$-base of seminorms for τ^-. That is to say that there exists a τ^--continuous seminorm q, for which there do not exist a τ-continuous seminorm p and a $\mu \in B_K^-$ such that q is p_μ-continuous (see Definition 3.4.38). For example, let $0 < r < 1$ and let $\mathrm{B}(0,r) = \{\lambda \in B_K \mid |\lambda| \le r\}$. Let $A = B_K/B(0,r)$. Let, as in Example 3.2.10, the norm $\|\ \|$ on A be defined by

$$\|\mu + B(0,r)\| = (|\mu| - r) \vee 0 \quad (\mu \in B_K).$$

Let τ be the $\|\ \|$-topology. We first prove that $\tau^- = \tau$. In fact, let $\varepsilon > 0$. Let $\lambda \in B_K^-$ such that $|\lambda|(r + \varepsilon) > r$. Then $\{x \in A \mid \|x\| < |\lambda|(r+\varepsilon) - r\}$ is open in the $\|\ \|$-topology and

$$\begin{aligned}
&\lambda^{-1}\{x \in A \mid \|x\| < |\lambda|(r+\varepsilon) - r\} = \\
&\lambda^{-1}\{\mu + \mathrm{B}(0,r) \mid |\mu| < |\lambda|(r+\varepsilon)\} = \{\nu + \mathrm{B}(0,r) \mid |\lambda\nu| < |\lambda|(r+\varepsilon)\} = \\
&\qquad \{\nu + \mathrm{B}(0,r) \mid |\nu| < r + \varepsilon\} = \{x \in A \mid \|x\| < \varepsilon\}.
\end{aligned}$$

We see that $\{x \in A \mid \|x\| < \varepsilon\}$ is τ^--open for all $\varepsilon > 0$. Hence the $\|\ \|$-topology is weaker than τ^-. This, together with Proposition 4.4.6, shows that τ^- equals the $\|\ \|$-topology. Thus, $\|\ \|$ is τ^--continuous. But let p be a τ-continuous seminorm on A and let $\mu \in B_K^-$. Then

$$\{x \in A \mid \|x\| < |\mu|^{-1}r - r\} \subset \operatorname{Ker} p_\mu,$$

whereas $\operatorname{Ker}\|\ \| = 0$. Hence, not $\operatorname{Ker}\|\ \| \subset \operatorname{Ker} p_\mu$. By using Proposition 3.3.11 we obtain that $\|\ \|$ is not p_μ-continuous.
We see that there do not exist a τ-continuous seminorm p and a $\mu \in B_K^-$ such that $\|\ \|$ is p_μ-continuous. Hence the collection

$$(p_\mu)_{\mu \in B_K^-,\ p\ \tau\text{-continuous seminorm on } A}$$

is not a $*$-base of seminorms for τ^-.

4.4.10 Proposition *Let (A, τ) be a locally convex B_K-module and let $(x_\alpha)_{\alpha\in I}$ be a net in A. Then:*

$$x_\alpha \xrightarrow{\tau^-} 0 \iff \lambda x_\alpha \xrightarrow{\tau} 0 \text{ for every } \lambda \in B_K^-$$

Proof: Let $\mathcal{P}$ be a collection of seminorms generating τ. Combining Theorem 4.4.8 and Proposition 3.4.5 we obtain that

$$x_\alpha \xrightarrow{\tau^-} 0 \iff p_\lambda(x_\alpha) \to 0 \text{ for all } p \in \mathcal{P} \text{ and all } \lambda \in B_K^- \iff$$
$$p(\lambda x_\alpha) \to 0 \text{ for all } p \in \mathcal{P} \text{ and all } \lambda \in B_K^- \iff \lambda x_\alpha \xrightarrow{\tau} 0 \text{ for all } \lambda \in B_K^-.$$

□

4.4.11 Remark If (A, τ) is a Hausdorff locally convex B_K-module, then (A, τ^-) need not be Hausdorff.
For example, let $A = B_K / B_K^-$ and let τ be the discrete topology on A. Then (A, τ) is a locally convex Hausdorff B_K-module.
Let V be a τ^--open submodule. There exists a $\lambda \in K$ with $|\lambda| > 1$ and a τ-open submodule U such that $\lambda U \subset V$. Then $A = \lambda\{0\} \subset \lambda U \subset V$ and therefore $V = A$.
Thus τ^- is the indiscrete topology on A and hence (A, τ^-) is not Hausdorff.

4.4.12 Proposition *Let (A, τ) be a Hausdorff locally convex B_K-module. Then: (A, τ^-) is Hausdorff $\iff$ A has no simple submodules.*
(For a the definition of a simple B_K-module see Definition 2.4.15.)

Proof: $\Rightarrow$) Let $x \in A$ with $x \neq 0$. Let V be a τ^--open submodule such that $x \notin V$. There exists a $\lambda \in K$ with $|\lambda| > 1$ and a τ-open submodule U such that $\lambda U \subset V$. Then $x \notin \lambda U$, which means that $\lambda^{-1}x \notin U$. Then in particular $\lambda^{-1}x \neq 0$. Hence, $\{0\} \subsetneqq \operatorname{co}\{\lambda^{-1}x\} \subsetneqq \operatorname{co}\{x\}$. That means that x can not be a member of any simple submodule of A. We may conclude that A has no simple submodules.
$\Leftarrow$) Let $x \in A$, $x \neq 0$. Then $\operatorname{co}\{x\}$ is not a simple submodule of A, hence there exists a $\lambda \in B_K^-$ such that $\lambda x \neq 0$. Let U be a τ-open submodule of A such that $\lambda x \notin U$. Then $\lambda^{-1}U$ is a τ^--open submodule and $x \notin \lambda U$.
We see that (A, τ^-) is Hausdorff. □

4.4.13 Remark Let (A, τ) be a locally convex B_K-module and let M be the union of all simple submodules of A. Then $M \subset V$ for every τ^--open submodule V of A.
In fact, if V is a τ^--open submodule of A then there exists a $\lambda \in K$, $|\lambda| > 1$ and a τ-open submodule U such that $\lambda U \subset V$. Then $M \subset \lambda\{0\} \subset \lambda U \subset V$.

Now we will provide two examples of locally convex B_K-modules where the minus topology and the initial topology do not coincide. The first example is a B_K-module of rank 1. The second one is a torsion free B_K-module.

4.4.14 Example Let $0 < r < 1$ and let $A = B_K/\mathrm{B}(0,r)$. Let the norm $\|\ \|$ on A be defined by

$$\|\lambda + \mathrm{B}(0,r)\| = (|\lambda| - r) \vee 0 \quad (\lambda \in B_K).$$

Let d be the discrete topology on A. We prove that d^- equals the $\|\ \|$-topology. As A contains no simple submodules we obtain, by Proposition 4.4.12, that d^- is Hausdorff. Furthermore, it is not hard to see that there does not exist a $\lambda \in K$, $|\lambda| > 1$ such that $\lambda\{0\} \subset \{0\}$. Hence, $\{0\}$ is not open in the d^--topology and therefore d^- is not discrete. From Proposition 4.1.5 we know that d and the $\|\ \|$-topology are the only locally convex Hausdorff topologies on A. Hence, d^- equals the $\|\ \|$-topology.

4.4.15 Example Let $A = B_K^{\mathbb{N}}$ and let the norm $\|\ \|$ on A be defined by

$$\|x\| = |x_0| \vee \sup_{i \geq 1} |x_i|^i \quad \left(x = (x_0, x_1, x_2, \ldots) \in B_K^{\mathbb{N}}\right)$$

Let τ be the $\|\ \|$-topology. Let σ be the product topology on A. We prove that $\tau^- = \sigma$. To this end we show the following.
Let $(x_\alpha)_{\alpha \in I}$ be a net in A. Then

$$x_\alpha \xrightarrow{\sigma} 0 \iff \lambda x_\alpha \xrightarrow{\tau} 0 \text{ for every } \lambda \in B_K^-.$$

$\Rightarrow$) Let $\lambda \in B_K^-$. Let $\varepsilon > 0$. Let $N \in \mathbb{N}$ such that $|\lambda|^N < \varepsilon$. There exists a $\gamma \in I$ such that $|(x_\alpha)_k| < \varepsilon$ for $\alpha > \gamma$ for all $k \in \{0, \ldots, N-1\}$. Then

$$\|\lambda x_\alpha\| = |\lambda x_\alpha|_0 \vee \max_{1 \leq k \leq N-1} |(\lambda x_\alpha)_k|^k \vee \sup_{k \geq N} |(\lambda x_\alpha)_k|^k \leq \varepsilon \vee \varepsilon \vee |\lambda|^N = \varepsilon.$$

We see that $\lambda x_\alpha \xrightarrow{\tau} 0$.
$\Leftarrow$) Let $n \in \mathbb{N}$. We prove that $|(x_\alpha)_n| \to 0$. For $n = 0$ we have that $|(x_\alpha)_0| \leq \frac{1}{\lambda}\|\lambda x_\alpha\|$ and $\|\lambda x_\alpha\| \to 0$. Hence also $|(x_\alpha)_0| \to 0$. Suppose $n \geq 1$. Let $\varepsilon > 0$. Let $\lambda \in K$ such that $1 > |\lambda|^n > \frac{1}{2}$. Then $\lambda x_\alpha \xrightarrow{\tau} 0$ and hence $\|\lambda x_\alpha\| < \frac{1}{2}\varepsilon^n$ for large α. In particular, $|\lambda|^n |(x_\alpha)_n|^n = |(\lambda x_\alpha)_n|^n < \frac{1}{2}\varepsilon^n$ for large α. Together with $|\lambda|^n > \frac{1}{2}$ this implies that $|(x_\alpha)_n| < \varepsilon$.
We see that $x_\alpha \xrightarrow{\sigma} 0$.
By using Proposition 4.4.10 we obtain that $\tau^- = \sigma$.
We have $\sigma \neq \tau$ which can be seen as follows. Let $e_0, e_1, e_2, \ldots$ be the canonical unit vectors of A. Then $e_n \xrightarrow{\sigma} 0$, but $\|e_n\| = 1$ for every $n \in \mathbb{N}$ and thus not $e_n \xrightarrow{\tau} 0$.

We conclude this part with a theorem about the metrizability of the minus topology.

4.4.16 Theorem *Let (A, τ) be a metrizable locally convex B_K-module such that A has no simple submodules. Then also (A, τ^-) is metrizable.*

Proof: By Theorem 3.4.15 there exists a norm $\| \ \|$ on A inducing τ. We may suppose that $\sup \| \ \| \leq 1$. Let $\lambda_1, \lambda_2, \ldots \in B_K$ be such that $|\lambda_1| < |\lambda_2| < \ldots$ and $\lim_{n\to\infty} |\lambda_n| = 1$. For all $n \geq 1$, let $\| \ \|_n$ be defined by

$$\|x\|_n = \|\lambda_n x\| \quad (x \in A).$$

By Theorem 4.4.8, $(\| \ \|_n)_{n\geq 1}$ is a countable collection of continuous seminorms generating τ^-. As A contains no simple submodules we have that for every $x \in A$ there exists an $n \in \mathbb{N}$ with $\lambda_n x \neq 0$. Then $\|x\|_n = \|\lambda_n x\| > 0$. Hence, $(\| \ \|_n)_{n\geq 1}$ is separating. Now apply Proposition 3.4.14. □

The Plus Topology

4.4.17 Definition Let (A, τ) be a locally convex B_K-module. A submodule V of A is called τ^+*-open* if for every $\lambda \in K$, $|\lambda| > 1$ the submodule λV is τ-open.
A subset W of A is called τ^+-open if for every $y \in W$ there exists a τ^+-open submodule V of A such that $y + V \subset W$.

4.4.18 Remark Like in Remark 4.4.2 we can make the following definition that is also suitable for locally convex B_K-modules over fields with a discrete valuation.
For a subset X of A and an $r > 0$ we define

$$rA = \bigcup\{\lambda A \mid \lambda \in K,\ |\lambda| \leq r\}.$$

A submodule of a locally convex B_K-module is called τ^+*-open* if for every $r > 1$ the submodule rV is τ-open.
A subset W of A is called τ^+-open if for every $y \in W$ there exists a τ^+-open submodule V such that $y + V \subset W$.
This definition is equivalent with the above one if $|K|$ is dense, and if $|K|$ is discrete, then $\tau^+ = \tau$.

4.4.19 Proposition *Let (A, τ) be a locally convex B_K-module. Then τ^+ is a locally convex topology.*

Proof: Let C be the collection of all submodules V of A for which λV is τ-open for every $\lambda \in K$ with $|\lambda| > 1$. Then C is a base of zero neighbourhoods of τ^+.
Let $V \in C$. Let $\lambda \in K$ with $|\lambda| > 1$. Then λV is τ-open and hence absorbing. Then also V is absorbing.
Let $n \in \mathbb{N}$ and $V_1, \ldots, V_n \in C$. Let $\lambda \in K$ with $|\lambda| > 1$. Then it is not hard to verify that

$$\lambda(V_1 \cap \ldots \cap V_n) = \lambda V_1 \cap \ldots \cap \lambda V_n.$$

The latter set is τ-open as λV_i is τ-open for every $i \in \{1, \ldots, n\}$.
Hence $V_1 \cap \ldots \cap V_n \in C$.
We see that C is a collection of absorbing submodules of A such that for every finite subcollection $\mathcal{F}$ of C there exists a $W \in C$ such that $W \subset \bigcap \mathcal{F}$.
Then τ^+ equals the C-topology and the latter is a locally convex topology according to Proposition 3.1.26. □

4.4.20 Proposition *Let (A, τ) and (B, σ) be locally convex B_K-modules. Let $\varphi : (A, \tau) \to (B, \sigma)$ be a continuous homomorphism. Then also the map $\varphi : (A, \tau^+) \to (B, \sigma^+)$ is continuous.*

Proof: Let V be a σ^+-open submodule of B. For every $\lambda \in K$, $|\lambda| > 1$ the set λV is σ-open and, by 2. of Proposition 2.1.20, $\lambda\varphi^{-1}(V) = \varphi^{-1}(\lambda V)$. The latter set is τ-open for every $\lambda \in K$, $|\lambda| > 1$ and hence $\varphi^{-1}(V)$ is τ^+-open. Thus, $\varphi : (A, \tau^+) \to (B, \sigma^+)$ is continuous. □

4.4.21 Remark Let (A, τ) be a locally convex B_K-module and let B be a submodule of A. As the inclusion map $(B, \tau|B) \to (A, \tau)$ is continuous we obtain, by the previous proposition, that $\tau^+|B \le (\tau|B)^+$.
But in general $(\tau|B)^+$ and $\tau^+|B$ need not to coincide.
For example, let $K = \mathbb{C}_p$ and let $A = B_K^{\mathbb{N}}$. Let τ be the product topology on A. Let $B = pA = \{x \in B_K^{\mathbb{N}} \mid |x_n| \le \frac{1}{p} \text{ for every } n \in \mathbb{N}\}$. Let

$$U = \{x \in B \mid |x_n| < \frac{1}{p} \text{ for all } n \in \mathbb{N}\}.$$

That U is $(\tau|B)^+$-open in B can be seen as follows. For every $\lambda \in K$, $|\lambda| > 1$ the set $\lambda U := \{x \in B \mid \lambda^{-1}x \in U\}$ equals B and hence λU is $\tau|B$-open for every $\lambda \in K$ with $|\lambda| > 1$.
The following observations show that U is not $\tau^+|B$-open. Let $e_0, e_1, e_2, \ldots$ be the canonical unit vectors in $B_K^{\mathbb{N}}$. Then $e_n \xrightarrow{\tau} 0$. It is not hard to verify that then $pe_n \xrightarrow{\tau^+} 0$ (see also Remark 4.4.28). Thus also $pe_n \to 0$ in B with respect to $\tau^+|B$. But $pe_n \in U$ for no $n \in \mathbb{N}$, hence U can not be $\tau^+|B$-open. We see that $\tau^+|B \lneq (\tau|B)^+$.

4.4.22 Proposition *Let (A, τ) be a locally convex B_K-module. Then:*

1. $\tau \le \tau^+$.

2. $\tau^{++} = \tau^+$.

Proof: 1. Let U be a τ-open submodule of A. Then λU is τ-open for every $\lambda \in K$ with $|\lambda| > 1$. Hence, U is τ^+-open.
2. Let V be a τ^{++}-open submodule of A. Let $\lambda \in K$ with $|\lambda| > 1$. Let $\mu \in K$ such that $1 < |\mu|^2 < |\lambda|$. Then μV is τ^+-open and hence $\mu^2 V$ is τ-open. Then also λV is τ-open, for $\mu^2 V \subset \lambda V$.
We see that every τ^{++}-open submodule is also τ^+-open and from 1. it follows that $\tau^{++} = \tau^+$. □

4.4.23 Corollary *If (A, τ) is a Hausdorff locally convex B_K-module, then also (A, τ^+) is Hausdorff.*

4.4.24 Proposition *Let (A, τ) be a locally convex B_K-module. Then*

1. *τ^+ is the strongest among all locally convex topologies σ with $\sigma^- = \tau^-$.*

2. *τ^- is the weakest among all locally convex topologies σ with $\sigma^+ = \tau^+$.*

Proof: **1.** First we observe that $\tau^{+-} = \tau^-$. In fact, clearly $\tau^- \leq \tau^{+-}$; we prove $\tau^{+-} \leq \tau^-$. To this end, let U be τ^{+-}-open submodule of A. Then there exists a $\mu \in K$ with $|\mu| > 1$ and a τ^+-open submodule V of A with $\mu V \subset U$. Let $\lambda \in K$ be such that $1 < |\lambda|^2 < |\mu|$. Then λV is τ-open and $\lambda(\lambda V) = \lambda^2 V \subset \mu V \subset U$ and hence U is τ^--open.
We see that $\tau^{+-} = \tau^-$.
Let σ be a locally convex topology on A such that $\sigma^- = \tau^-$.
Let U be a σ-open submodule and let $\lambda \in K$ with $|\lambda| > 1$. Then λU is σ^--open and hence τ^--open. Then λU is also τ-open since $\tau^- \leq \tau$.
We see that for every $\lambda \in K$ with $|\lambda| > 1$ the submodule λU is τ-open which means that U is τ^+-open.
We obtain that $\sigma \leq \tau^+$.
2. This part of the proof is in the same spirit as **1.** and is left to the reader.
□

Next we provide two examples in which we see that the plus topology and the initial topology do not coincide.

4.4.25 Example Let $0 < r < 1$. Let $A = B_K/\mathrm{B}(0,r)$. Let d be the discrete topology on A and let τ be the $\|\ \|$-topology, where $\|\ \|$ on A is defined by

$$\|\lambda + \mathrm{B}(0,r)\| = (|\lambda| - r) \vee 0 \quad (\lambda \in B_K).$$

In Example 4.4.14 we have seen that $d^- = \tau$ on A. By using Proposition 4.4.24 we obtain that $\tau^+ \geq d$ and hence, $\tau^+ = d \neq \tau$.

From Example 4.4.15 we can see that in general, even on torsion free B_K-modules the plus topology and the initial topology do not coincide:

4.4.26 Example We consider again the locally convex B_K-module $(B_K^{\mathbb{N}}, \|\ \|)$, where $\|\ \|$ is defined by

$$\|(\lambda_0, \lambda_1, \lambda_2, \ldots)\| = |\lambda_0| \vee \sup_{n \geq 1} |\lambda_n|^n \quad ((\lambda_0, \lambda_1, \lambda_2, \ldots) \in B_K^{\mathbb{N}})).$$

Let τ be the $\|\ \|$-topology. In Example 4.4.15 we have seen that $\tau^- = \sigma$ and $\tau \neq \sigma$, where σ is the product topology on $B_K^{\mathbb{N}}$. Combining this with Proposition 4.4.24 we obtain that $\tau \leq \sigma^+$ and hence $\sigma^+ \neq \sigma$.
It is not true that $\sigma^+ = \tau$. We will end this section with a more detailed study of the plus topology of the product topology on $B_K^{\mathbb{N}}$.

4.4.27 Proposition *Let (A, τ) be a locally convex B_K-module. Let $(x_\alpha)_{\alpha \in I}$ be a net in A. Then:*

$$\lambda x_\alpha \xrightarrow{\tau^-} 0 \text{ for every } \lambda \in B_K^- \iff \lambda x_\alpha \xrightarrow{\tau} 0 \text{ for every } \lambda \in B_K^- \iff$$
$$\lambda x_\alpha \xrightarrow{\tau^+} 0 \text{ for every } \lambda \in B_K^-.$$

Proof: By using Proposition 4.4.10, Proposition 4.4.24 and Proposition 4.4.6 we obtain the following equivalences.

$$\begin{aligned}
&\lambda x_\alpha \xrightarrow{\tau^-} 0 \text{ for every } \lambda \in B_K^- \iff x_\alpha \xrightarrow{\tau^{--}} 0 \iff x_\alpha \xrightarrow{\tau^-} 0 \iff \\
&\lambda x_\alpha \xrightarrow{\tau} 0 \text{ for every } \lambda \in B_K^- \iff x_\alpha \xrightarrow{\tau^-} 0 \iff x_\alpha \xrightarrow{\tau^{+-}} 0 \iff \\
&\lambda x_\alpha \xrightarrow{\tau^+} 0 \text{ for every } \lambda \in B_K^-.
\end{aligned}$$

□

4.4.28 Remark Let (A, τ) be a locally convex B_K-module and let $(x_\alpha)_{\alpha \in I}$ be a net in A. The previous proposition implies that

$$x_\alpha \xrightarrow{\tau} 0 \Rightarrow \lambda x_\alpha \xrightarrow{\tau^+} 0 \text{ for every } \lambda \in B_K^-.$$

But the equivalence

$$x_\alpha \xrightarrow{\tau} 0 \iff \lambda x_\alpha \xrightarrow{\tau^+} 0 \text{ for every } \lambda \in B_K^-,$$

like in Proposition 4.4.10 is in general not true, because that should imply that

$$x_\alpha \xrightarrow{\tau^-} 0 \iff x_\alpha \xrightarrow{\tau} 0$$

(see the proof of the previous proposition) and hence $\tau^- = \tau$. In Example 4.4.15 we have seen that $\tau^- = \tau$ is in general not true.

4.4.29 Theorem *Let (A, τ) be a locally convex B_K-module and let p be a seminorm on A. Then:*

$$p \text{ is } \tau^+\text{-continuous} \iff p_\lambda \text{ is } \tau\text{-continuous for every } \lambda \in B_K^-.$$

Proof: $\Rightarrow$) Suppose that p is τ^+-continuous. Let $\lambda \in B_K^-$ and $\varepsilon > 0$. Then

$$\{x \in A \mid p_\lambda(x) < \varepsilon\} = \{x \in A \mid p(\lambda x) < \varepsilon\} = \lambda^{-1}\{x \in A \mid p(x) < \varepsilon\}.$$

And the latter set is τ-open and hence so is $\{x \in A \mid p_\lambda(x) < \varepsilon\}$.
From Proposition 3.3.7 we obtain that p_λ is τ-continuous.
$\Leftarrow$) Suppose that p_λ is τ-continuous for every $\lambda \in B_K^-$. Let $\varepsilon > 0$. Let $\mu \in K$ with $|\mu| > 1$. Then

$$\mu\{x \in A \mid p(x) < \varepsilon\} = \{x \in A \mid p(\mu^{-1}x) < \varepsilon\} = \{x \in A \mid p_{\mu^{-1}}(x) < \varepsilon\}.$$

And the latter set is τ-open.
Thus, for every $\mu \in K$ with $|\mu| > 1$ the set $\mu\{x \in A \mid p(x) < \varepsilon\}$ is τ-open and hence $\{x \in A \mid p(x) < \varepsilon\}$ is τ^+-open.
We see that p is τ^+-continuous. □

The next proposition is about the quotient topology of the plus topology.

4.4.30 Proposition *Let (A, τ) be a locally convex B_K-module. Let B be a submodule of A. Let σ be the quotient topology on A/B. Then the quotient topology of τ^+ on A/B equals σ^+.*

Proof: 1. Let V be an open submodule in the quotient topology of τ^+. Then there exists a τ^+-open submodule U of A such that $\pi(U) = V$. (Here $\pi : A \to A/B$ is the quotient map.)
Let $\lambda \in K$ with $|\lambda| > 1$. Then λU is τ-open and hence $\pi(\lambda U)$ is σ-open. Now $\pi(\lambda U) \subset \lambda\pi(U)$ and hence $\lambda\pi(U)$ is σ-open.
We obtain that $V = \pi(U)$ is σ^+-open
2. Let V be a σ^+-open submodule of A/B. Let $\lambda \in K$ with $|\lambda| > 1$.
Then λV is σ-open and hence $\lambda\pi^{-1}(V) = \pi^{-1}(\lambda V)$ is τ-open in A.
We obtain that $\pi^{-1}(V)$ is τ^+-open and hence $V = \pi(\pi^{-1}(V))$ is open in the quotient topology of τ^+. □

4.4.31 Remark Proposition 4.4.30 does not remain true if we replace τ^+ and σ^+ by τ^- and σ^- respectively.
For example, let $(A, \tau) = (B_K, |\ |)$. Let $r \in (0,1)$. Then $\mathrm{B}(0,r)$ is a submodule of A.
Let $\|\ \|$ on $A/\mathrm{B}(0,r)$ be defined by $\|\lambda + \mathrm{B}(0,r)\| = (|\lambda| - r) \vee 0 \quad (\lambda \in B_K)$.
Let $\pi : A \to A/\mathrm{B}(0,r)$ be the quotient map.
From Proposition 4.4.12 we obtain that τ^- is Hausdorff. By Proposition 4.1.4 τ is the only locally convex Hausdorff topology on A and hence $\tau^- = \tau$.
Now the quotient topology of τ^- is equal to the quotient topology of the $|\ |$-topology which is obviously the discrete topology d.
Of course, the quotient topology of τ also equals d, and d^- is the $\|\ \|$-topology, which we have already seen in Example 4.4.14.
Since the $\|\ \|$-topology is not equal to the discrete topology we see that the quotient topology of τ^- and the minus topology of the quotient topology of τ do not coincide.

Now we want to mention some things on the metrizability of the plus topology. First we observe the following. If A is a B_K-module for which $\lambda A = \{0\}$ for every $\lambda \in B_K^-$ then τ^+ equals the discrete topology for every locally convex topology τ on A.
One might think that on such a B_K-module the discrete topology is the only locally convex topology. But that is not true.
For example, let $A = (B_K/B_K^-)^{\mathbb{N}}$. Then $\lambda A = \{0\}$ for every $\lambda \in B_K^-$. It is not hard to see that the product topology on A induced by the discrete topology on B_K/B_K^- is not discrete.
More general, if (A, τ) is a locally convex B_K-module such that there exists a τ-open submodule U of A such that $\lambda U = \{0\}$ for every $\lambda \in B_K^-$, then τ^+ is discrete and hence metrizable.

In general we can not say a lot about the metrizability of the plus topology. For Hausdorff locally convex B_K-modules we can prove the following.

4.4.32 Proposition *Let (A, τ) be a Hausdorff locally convex B_K-module such that τ^+ is metrizable. Then there exists a $\lambda \in B_K^-$ such that $\overline{\lambda A}$ is τ^+-open. (Here $\overline{\lambda A}$ is the closure of λA with respect to τ.)*

Proof: We start the proof with the observation that $\lambda A^- = A$, for every $\lambda \in K$, $|\lambda| > 1$ (recall that $A^- = \bigcup\{\lambda A \mid \lambda \in B_K^-\}$). Hence, A^- is τ^+-open.
Suppose that $\overline{\lambda A}$ is τ^+-open for no $\lambda \in B_K^-$. Let $U_1 \supset U_2 \supset U_3 \supset \dots$ be a base of zero neighbourhoods of τ^+. As A^- is τ^+-open we may assume that $U_n \subset A^-$ for every $n \geq 1$.
Let $\lambda_1 \in B_K^-$. Then $\overline{\lambda_1 A}$ is not τ^+-open and hence not $U_1 \subset \overline{\lambda_1 A}$. Thus there exists an $x_1 \in U_1$ with $x_1 \notin \overline{\lambda_1 A}$. As $U_1 \subset A^-$ there exists a $\lambda_2 \in B_K^-$ such that $x_1 \in \lambda_2 A$. Then $|\lambda_1| < |\lambda_2|$.
Now $\overline{\lambda_2 A}$ is not τ^+-open and hence not $U_2 \subset \overline{\lambda_2 A}$. Thus there exists an $x_2 \in U_2$ with $x_2 \notin \overline{\lambda_2 A}$. As $U_2 \subset A^-$ there exists a $\lambda_3 \in B_K^-$ such that $x_2 \in \lambda_3 A$. Then $|\lambda_2| < |\lambda_3|$.
Now also $\overline{\lambda_3 A}$ is not τ^+-open and hence not $U_3 \subset \overline{\lambda_3 A}$. Thus there exists an $x_3 \in U_3$ such that $x_3 \notin \overline{\lambda_3 A}$.
Continuing this way we find $\lambda_1, \lambda_2, \lambda_3, \dots \in B_K^-$, $|\lambda_1| < |\lambda_2| < |\lambda_3| < \dots$ and $x_1, x_2, x_3, \dots \in A$ such that $x_n \in U_n \cap \lambda_{n+1} A$ and $x_n \notin \overline{\lambda_n A}$ for all $n \geq 1$. We can choose the λ_n's in such a way that $|\lambda_n| > \frac{n}{n+1}$ for every $n \geq 1$, and hence $\lim_{n\to\infty} |\lambda_n| = 1$.
We obtain $x_n \xrightarrow{\tau^+} 0$. By Theorem 3.4.22 there exist τ-continuous seminorms $p_1, p_2, p_3, \dots$ on A such that $p_n \leq 1$, $p_n | \overline{\lambda_n A} = 0$ and $p_n(x_n) = 1$ for all $n \geq 1$. It is not hard to verify that the map p on $\bigcup_{n\geq 1} \overline{\lambda_n A}$ defined by

$$p(x) = \sup_{n\geq 1} p_n(x) \quad (x \in \bigcup_{n\geq 1} \overline{\lambda_n A})$$

is a seminorm, that $\sup p = 1$ and that

$$\begin{cases} p|\overline{\lambda_1 A} = 0 \\ p|\overline{\lambda_n A} = (p_1 \vee p_2 \vee \dots \vee p_n)|\overline{\lambda_n A} \quad (n \geq 2). \end{cases}$$

By Theorem 3.3.4 there exists a seminorm q on A with $\sup q = 1$ such that $q | \bigcup_{n\geq 1} \overline{\lambda_n A} = p$.
Let $n \geq 1$. Then $q|\overline{\lambda_n A} = (p_1 \vee \dots \vee p_n)|\overline{\lambda_n A}$ and as $p_1, \dots, p_n$ are τ-continuous we obtain that $q|\overline{\lambda_n A}$ is τ-continuous. In particular, $q|\lambda_n A$ is τ-continuous. Then also q_{λ_n} is τ-continuous, since $q_{\lambda_n} = (q|\overline{\lambda_n A}) \circ M_{\lambda_n}$. (Here $M_{\lambda_n} : A \to \lambda_n A$ is defined by $M_{\lambda_n}(x) = \lambda_n x \quad (x \in A)$, which is, by 3. of Proposition 3.1.7, a continuous map.)
We see that q_{λ_n} is τ-continuous for every $n \geq 1$. From Theorem 4.4.29 we obtain that q is τ^+-continuous.
On the other hand, $x_n \xrightarrow{\tau^+} 0$ but $q(x_n) \geq p_n(x_n) = 1$ for $n \geq 1$, a contradiction.
We see that there exists a $\lambda \in B_K^-$ such that $\overline{\lambda A}$ is τ^+-open. □

4.4.33 Remark From this proof we obtain the following. Let (A, τ) be a bounded Hausdorff locally convex B_K-module. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A such that $x_n \xrightarrow{\tau^+} 0$. Then there exists a $\lambda \in B_K^-$ such that $x_n \in \overline{\lambda A}$ for large n (here $\overline{\lambda A}$ is the closure of λA with respect to τ).

In general it is not true that the plus topology of a metrizable locally convex topology on a B_K-module A is metrizable. In Corollary 4.4.54 we will see that the plus topology of the product topology of $B_K^{\mathbb{N}}$ is not metrizable, whereas the product topology itself is metrizable.

The Inductive Limit Topology

4.4.34 Definition Let (A, τ) be a locally convex B_K-module. A seminorm p on A is called *τ^{ind}-continuous* if for every $\lambda \in B_K^-$ the seminorm $p|\lambda A$ is continuous with respect to $\tau|\lambda A$.
τ^{ind} is the topology generated by the τ^{ind}-continuous seminorms.
The τ^{ind}-topology is called the *inductive limit topology.*

4.4.35 Proposition *τ^{ind} is a locally convex topology.*

Proof: τ^{ind} is generated by a collection of seminorms. □

One could expect now that the following will be proved.
Let (A, τ) and (B, σ) be locally convex B_K-modules. Let $\varphi : (A, \tau) \to (B, \sigma)$ be a continuous homomorphism. Then also $\varphi : (A, \tau^{\text{ind}}) \to (B, \sigma^{\text{ind}})$ is continuous.
We will do this later on: see Proposition 4.4.42.

4.4.36 Proposition *Let (A, τ) be a locally convex B_K-module and let U be a submodule of A. Then:*
U is τ^{ind}-open $\iff$ $U \cap \lambda A$ is open in λA with respect to $\tau|\lambda A$ for every $\lambda \in B_K^-$.

Proof: $\Rightarrow$) Suppose that U is τ^{ind}-open. Then p_U is a τ^{ind}-continuous seminorm and hence $p_U|\lambda A$ is $\tau|\lambda A$-continuous for every $\lambda \in B_K^-$.
Then $U \cap \lambda A = \{x \in \lambda A \mid (p_U|\lambda A)(x) < 1\}$ is $\tau|\lambda A$-open for all $\lambda \in B_K^-$.
$\Leftarrow$) Suppose that $U \cap \lambda A$ is $\tau|\lambda A$-open for every $\lambda \in B_K^-$.
Let $V = \bigcup_{|\lambda|<1}(U \cap \lambda A)$. Then $U \cap \lambda A \subset V$ and hence $p_V|\lambda A = 0$ on $U \cap \lambda A$ for all $\lambda \in B_K^-$.
Let $\lambda \in B_K^-$. Then $U \cap \lambda A \subset \{x \in \lambda A \mid p_V(x) < \varepsilon\}$ for all $\varepsilon > 0$ and hence the latter set is $\tau|\lambda A$-open for all $\varepsilon > 0$. This means that p_V is $\tau|\lambda A$-continuous. We see that p_V is τ^{ind}-continuous.
Now

$$\{x \in A \mid p_V(x) < 1\} = \{x \in A \mid x \in \bigcup_{|\lambda|<1} (U \cap \lambda A)\} = \bigcup_{|\lambda|<1} (U \cap \lambda A) \subset U$$

and hence U is τ^{ind}-open. □

4.4.37 Proposition $\tau \leq \tau^{\text{ind}} \leq \tau^+$.

Proof: We only prove $\tau^{\text{ind}} \leq \tau^+$.
Let V be a τ^{ind}-open submodule of A. Let $\lambda \in K$ with $|\lambda| > 1$. The map

$S_{\lambda^{-1}} : (A, \tau) \to (\lambda^{-1}A, \tau|\lambda^{-1}A)$ defined by $S_{\lambda^{-1}}(x) = \lambda^{-1}x \quad (x \in A)$ is continuous and

$$\lambda V = \{x \in A \mid \lambda^{-1}x \in V\} = \{x \in A \mid \lambda^{-1}x \in V \cap \lambda^{-1}A\} = S_{\lambda^{-1}}^{-1}(V \cap \lambda^{-1}A).$$

The latter set is τ-open and hence so is λV.
We obtain that V is τ^+-open. □

4.4.38 Proposition *Let (A, τ) be a locally convex B_K-module. Then $(\tau^{\text{ind}})^{\text{ind}} = \tau^{\text{ind}}$.*

Proof: Let p be a $(\tau^{\text{ind}})^{\text{ind}}$-continuous seminorm. Let $\mu \in B_K^-$. Let $\lambda \in B_K^-$ such that $|\mu| \leq |\lambda|^2$. Then $p|\lambda A$ is $\tau^{\text{ind}}|\lambda A$-continuous on λA. Now $p|\lambda(\lambda A)$ is $\tau|\lambda^2 A$-continuous on $\lambda^2 A (= \lambda(\lambda A))$. As $\mu A \subset \lambda^2 A$ we obtain that p is $\tau|\mu A$-continuous on μA.
We see that every $(\tau^{\text{ind}})^{\text{ind}}$-continuous seminorm on A is also τ^{ind}-continuous. Thus, $(\tau^{\text{ind}})^{\text{ind}} \leq \tau^{\text{ind}}$. From the previous theorem we obtain $(\tau^{\text{ind}})^{\text{ind}} = \tau^{\text{ind}}$. □

4.4.39 Theorem *Let (A, τ) be a locally convex B_K-module. Let $(x_\alpha)_{\alpha \in I}$ be a net in A with the following properties.*

1) *There exists a $\lambda \in B_K^-$ such that $x_\alpha \in \lambda A$ for large α.*

2) *$x_\alpha \to 0$ with respect to τ.*

Then $x_\alpha \to 0$ with respect to τ^{ind}.

Proof: Let p be a τ^{ind}-continuous seminorm on A. Let $\lambda \in B_K^-$ such that $x_\alpha \in \lambda A$ for large α.
Now $p|\lambda A$ is continuous with respect to $\tau|\lambda A$ and by Theorem 3.4.20 there exists a τ-continuous seminorm q on A such that $q|\lambda A = p|\lambda A$.
Then $q(x_\alpha) \to 0$ and $x_\alpha \in \lambda A$ for large α thus $p(x_\alpha) = q(x_\alpha)$ for large α. Hence, also $p(x_\alpha) \to 0$.
We see that $p(x_\alpha) \to 0$ for every τ^{ind}-continuous seminorm p and hence $x_\alpha \xrightarrow{\tau^{\text{ind}}} 0$. □

4.4.40 Remark The converse of the above theorem is not true: see Example 4.4.55.

4.4.41 Theorem *Let (A, τ) be a locally convex torsion free B_K-module such that (A, τ) is topologically embeddable in a locally convex K-vector space (E, σ). Then $\tau^{\text{ind}} = \tau^+$.*

Proof: Let $\lambda \in B_K^- \setminus \{0\}$. Let the map $M_\lambda : (E, \sigma) \to (E, \sigma)$ be defined by $M_\lambda(x) = \lambda x \quad (x \in E)$. Then M_λ is a homeomorphism. This implies that also the map S_λ from (A, τ) to $(\lambda A, \tau|\lambda A)$ defined by $S_\lambda(x) = \lambda x \quad (x \in A)$ is a homeomorphism.
Let U be a τ^+-open submodule of A. Then $\lambda^{-1}U$ is τ-open in A and hence $U \cap \lambda A = \lambda(\lambda^{-1}U) = S_\lambda(\lambda^{-1}U)$ is $\tau|\lambda A$-open in λA.

We see that $U \cap \lambda A$ is open in λA for every $\lambda \in B_K^-$. By using Proposition 4.4.36 we obtain that U is τ^{ind}-open.
Hence, $\tau^+ \le \tau^{\text{ind}}$ and from Proposition 4.4.37 it follows that $\tau^{\text{ind}} = \tau^+$. □

4.4.42 Proposition *Let (A, τ) and (B, σ) be locally convex B_K-modules. Let $\varphi : (A, \tau) \to (B, \sigma)$ be a continuous homomorphism. Then the map $\varphi : (A, \tau^{\text{ind}}) \to (B, \sigma^{\text{ind}})$ is also continuous.*

Proof: Let q be a τ^{ind}-continuous seminorm on B. We prove that $q \circ \varphi$ is a τ^{ind}-continuous seminorm on A. To this end let $\lambda \in B_K^-$. Suppose $(x_\alpha)_{\alpha \in I}$ is a net in λA such that $x_\alpha \xrightarrow{\tau} 0$. Then there exists a net $(y_\alpha)_{\alpha \in I}$ in A such that $x_\alpha = \lambda y_\alpha \quad (\alpha \in I)$. Then $\varphi(x_\alpha) = \lambda \varphi(y_\alpha) \in \lambda B$ for all $\alpha \in I$. Furthermore, $\varphi(x_\alpha) \xrightarrow{\sigma} 0$ in B since $\varphi : (A, \tau) \to (B, \sigma)$ is continuous. By using Theorem 4.4.39 we obtain $\varphi(x_\alpha) \xrightarrow{\sigma^{\text{ind}}} 0$. Since q is τ^{ind}-continuous this implies that $q \circ \varphi(x_\alpha) = q(\varphi(x_\alpha)) \to 0$.
We see that $q \circ \varphi | \lambda A$ is $\tau | \lambda A$-continuous. This is true for all $\lambda \in B_K^-$ and hence $q \circ \varphi$ is τ^{ind} continuous.
We now prove that $\varphi : (A, \tau^{\text{ind}}) \to (B, \sigma^{\text{ind}})$ is continuous. To this end let U be a σ^{ind}-open submodule of B. Then p_U is a σ^{ind}-continuous seminorm on B. Thus, $p_U \circ \varphi$ is a τ^{ind}-continuous seminorm on A. Now the submodule $\{x \in A \mid p_U \circ \varphi(x) < 1\}$ is τ^{ind}-open in A and

$$\varphi(\{x \in A \mid p_U \circ \varphi(x) < 1\}) \subset \{y \in B \mid p_U(x) < 1\} = U.$$

Hence, $\varphi^{-1}(U)$ is τ^{ind}-open. □

4.4.43 Remark Let (A, τ) be a locally convex B_K-module and let B be a submodule of A. Then $(\tau | B)^{\text{ind}}$ and $\tau^{\text{ind}} | B$ in general do not coincide.
In the example in Remark 4.4.21 we have seen that, for the product topology τ on $B_K^{\mathbb{N}}$, $(\tau | B)^+$ and $\tau^+ | B$ do not coincide. Since $(B_K^{\mathbb{N}}, \tau)$ is topologically embeddable in a locally convex K-vector space the above assertion follows from Theorem 4.4.41.

Now we will give some examples of locally convex B_K-modules (A, τ) for which we determine τ^-, τ^+ and τ^{ind}.

4.4.44 Example Let $r \in (0, 1)$ and let $\mathrm{B}(0, r) = \{\lambda \in B_K \mid |\lambda| \le r\}$. Let $A = B_K / \mathrm{B}(0, r)$. Let τ be the $\| \ \|$-topology, where $\| \ \|$ on A is defined by

$$\|\lambda + \mathrm{B}(0, r)\| = (|\lambda| - r) \vee 0 \quad (\lambda \in B_K).$$

Let d be the discrete topology on A. In the example in Remark 4.4.9 we have already seen that $\tau^- = \tau$. In Example 4.4.25 we have seen that $\tau^+ = d$. Furthermore $\tau^{\text{ind}} = \tau$, for let U be a τ^{ind}-open submodule. We prove that U is τ-open. To this end let $\lambda \in B_K^-$ with $|\lambda| > r$. Then $U \cap \lambda A$ is $\tau | \lambda A$-open and hence there exists a τ-open submodule V such that $V \cap \lambda A = U \cap \lambda A$. There exists an $s > 0$ such that $\{x \in A \mid \|x\| < s\} \subset V$. We may suppose that $s < |\lambda| - r$. Then

$$\{x \in A \mid \|x\| < s\} = \{\mu + \mathrm{B}(0, r) \mid |\mu| < s + r\} \subset$$
$$\{\mu + \mathrm{B}(0, r) \mid |\mu| < |\lambda|\} \subset \lambda A.$$

Thus $\{x \in A \mid \|x\| < s\} \subset V \cap \lambda A = U \cap \lambda A \subset U$ and hence U is τ-open.
We see that τ is a locally convex topology on A with $\tau^- = \tau = \tau^{\mathrm{ind}} \subsetneq \tau^+$.
Furthermore, d is a locally convex topology on A with $d^- \subsetneq d = d^{\mathrm{ind}} = d^+$.

4.4.45 Example Let $r \in |B_K^-| \setminus \{0\}$. Let $\mathrm{B}(0,r^-) = \{\lambda \in B_K \mid |\lambda| < r\}$. Let $A = B_K/\mathrm{B}(0,r^-)$ and let τ be the discrete topology on A (according to Proposition 4.1.5 τ is the only Hausdorff locally convex topology on A).
Then of course $\tau^+ = \tau^{\mathrm{ind}} = \tau$.
Let $p : A \to [0,\infty)$ be defined by

$$p(\lambda + \mathrm{B}(0,r^-)) = (|\lambda| - r) \vee 0 \quad (\lambda \in B_K).$$

Then p is a seminorm on A and the p-topology equals τ^-.
For let U be a τ^--open submodule of A. Then there exists a τ-open submodule V and a $\lambda \in K$ with $|\lambda| > 1$ such that $\lambda V \subset U$. Then

$$\{x \in A \mid p(x) < |\lambda| r - r\} = \{\mu + \mathrm{B}(0,r^-) \mid |\mu| < |\lambda| r\} =$$
$$\{\mu + \mathrm{B}(0,r^-) \mid |\lambda^{-1}\mu| < r\} = \lambda\{0\} \subset \lambda V \subset U$$

and hence U is open in the p-topology.
Let U be a p-open submodule of A. There exists an $\varepsilon > 0$ such that $\{x \in A \mid p(x) < \varepsilon\} \subset U$. Let $\lambda \in K$, $|\lambda| > 1$ such that $|\lambda| r < r + \varepsilon$. Then

$$\lambda\{0\} = \{x \in A \mid \lambda^{-1}x = 0\} = \{\mu + \mathrm{B}(0,r^-) \mid |\lambda^{-1}\mu| < r\} =$$
$$\{\mu + \mathrm{B}(0,r^-) \mid |\mu| < |\lambda| r\} = \{x \in A \mid p(x) < |\lambda| r - r\}$$

and $\{x \in A \mid p(x) < |\lambda| r - r\} \subset \{x \in A \mid p(x) < \varepsilon\} \subset U$. Now $\{0\}$ is a τ-open submodule and hence U is τ^--open.
We see that τ^- is equal to the p-topology. Hence, τ^- is not Hausdorff and $\{\mu + \mathrm{B}(0,r^-) \mid |\mu| \le r\}$ is the simple submodule of A.
Thus, τ is a locally convex topology on A for which $\tau^- \subsetneq \tau = \tau^{\mathrm{ind}} = \tau^+$.

The following example is a torsion free Hausdorff locally convex B_K-module (A, τ) for which τ^+ and τ^{ind} do not coincide.

4.4.46 Example Let $A = B_K^{\mathbb{N}}$.
For every $n \in \mathbb{N}$ let $p_n : A \to [0,\infty)$ be defined by

$$p_n((\lambda_0,\lambda_1,\lambda_2,\dots)) = |\lambda_n| \quad \left((\lambda_0,\lambda_1,\lambda_2,\dots) \in B_K^{\mathbb{N}}\right).$$

Let $q : A \to [0,\infty)$ be defined by

$$q((\lambda_0,\lambda_1,\lambda_2,\dots)) = \sup_{n\in\mathbb{N}} \left((|\lambda_n| - \frac{1}{2}) \vee 0\right).$$

Let $\mathcal{P} = \{q, p_0, p_1, p_2, \dots\}$ and let τ be the $\mathcal{P}$-topology.
Let $r : A \to [0,\infty)$ be defined by

$$r((\lambda_0,\lambda_1,\lambda_2,\dots)) = \begin{cases} 1 & \text{if } |\lambda_n| \ge \frac{1}{2} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{if } |\lambda_n| < \frac{1}{2} \text{ for all } n \in \mathbb{N}. \end{cases}$$

Then r is a τ^+-continuous seminorm, which can be seen as follows.
Let $\lambda \in B_K^-$. Let $(x_\alpha)_{\alpha\in I}$ be a net in A such that $x_\alpha \xrightarrow{\tau} 0$. Then $q(x_\alpha) \to 0$ and hence $\sup_{n\in\mathbb{N}}((|(x_\alpha)_n| - \frac{1}{2}) \vee 0) < \frac{1}{2}|\lambda|^{-1} - \frac{1}{2}$ for large α. Then $|(x_\alpha)_n| < \frac{1}{2}|\lambda|^{-1}$ $(n \in \mathbb{N})$ for large α. Thus $r_\lambda(x_\alpha) = r(\lambda x_\alpha) = 0$ for large α and hence $r_\lambda(x_\alpha) \xrightarrow{\tau} 0$. We see that r_λ is τ-continuous for every $\lambda \in B_K^-$ and hence r is τ^+-continuous. But we claim that r is not τ^{ind}-continuous.
In fact, let $\lambda \in B_K$ with $\frac{1}{2} < |\lambda| < 1$. Let $\lambda_0, \lambda_1, \lambda_2, \ldots \in B_K$ such that $|\lambda| = |\lambda_0| > |\lambda_1| > |\lambda_2| > \ldots$ and $\frac{1}{2} < |\lambda_n| < \frac{1}{2} + \frac{1}{n}$ $(n \geq 1)$. Let

$$x_n = \lambda_n e_n = (0, \ldots, 0, \underbrace{\lambda_n}_{n^{\text{th}}\text{place}}, 0, \ldots) \quad (n \in \mathbb{N}).$$

Then $p_k(x_n) \to 0$ for all $k \in \mathbb{N}$, since $p_k(x_n) = 0$ for all $n > k$. Now $q(x_n) = |\lambda_n| - \frac{1}{2} < \frac{1}{n}$ and hence also $q(x_n) \to 0$. Thus $x_n \xrightarrow{\tau} 0$.
Furthermore, $x_n \in \lambda A$ for all $n \in \mathbb{N}$ and hence $x_n \xrightarrow{\tau^{\text{ind}}} 0$ (by Theorem 4.4.39). Now $(r|\lambda A)(x_n) = r(x_n) = 1$ for each $n \in \mathbb{N}$. Hence, it is not true that $(r|\lambda A)(x_n) \xrightarrow{\tau} 0$. That means that $r|\lambda A$ is not τ-continuous and thus r is not τ^{ind}-continuous.
We see that (A, τ) is a torsion free Hausdorff locally convex B_K-module and $\tau^{\text{ind}} \neq \tau^+$.

We will end this section with discussing the plus topology of the product topology on $B_K^{\mathbb{N}}$.

An Example: The Plus Topology of the Product topology on $B_K^{\mathbb{N}}$

We consider the B_K-module $B_K^{\mathbb{N}}$ provided with the product topology which we will denote τ. Then the collection

$$\left(\{x \in B_K^{\mathbb{N}} \mid |x_0| < \varepsilon, \ldots, |x_n| < \varepsilon\}\right)_{n\in\mathbb{N},\ \varepsilon>0}$$

is a base of zero neighbourhoods of τ.
τ is also generated by the norm $\|\ \|$ which is defined by

$$\|x\| = \max_{n\in\mathbb{N}} \frac{1}{n}|x_n| \quad \left(x = (x_0, x_1, x_2, \ldots) \in B_K^{\mathbb{N}}\right).$$

4.4.47 Proposition $\tau^+ = \tau^{\text{ind}}$.

Proof: τ is induced by the product topology on $K^{\mathbb{N}}$, which is a locally convex vector space topology on $K^{\mathbb{N}}$. From Theorem 4.4.41 we obtain that $\tau^+ = \tau^{\text{ind}}$.
□

4.4.48 Proposition *Let $x^1, x^2, x^3, \ldots \in B_K^{\mathbb{N}}$. Then*

$x^n \xrightarrow{\tau^+} 0 \iff$ 1. *There exists a $\lambda \in B_K^-$ such that $x^n \in \lambda B_K^{\mathbb{N}}$ for large n.*

2. $x^n \xrightarrow{\tau} 0$.

Proof: $\Rightarrow$) From Remark 4.4.33 we obtain that there exists a $\lambda \in B_K^-$ such that $x_n \in \overline{\lambda B_K^{\mathbb{N}}}$ for large n. As $\overline{\lambda B_K^{\mathbb{N}}} = \lambda B_K^{\mathbb{N}}$ we obtain 1. Obviously we have 2. since $\tau \le \tau^+$.
$\Leftarrow$) We have seen that $\tau^{\mathrm{ind}} = \tau^+$. Now apply Theorem 4.4.39. □

4.4.49 Proposition *The norm p on $B_K^{\mathbb{N}}$ defined by*

$$p(x) = |x_0| \vee \max_{n \ge 1} |x_n|^n \quad \left(x = (x_0, x_1, x_2, \ldots) \in B_K^{\mathbb{N}}\right)$$

is τ^+-continuous but not τ-continuous.

Proof: It is not hard to prove that p_λ is τ-continuous for every $\lambda \in B_K^-$. By Theorem 4.4.29 p is τ^+-continuous.
Let $e_0, e_1, e_2, \ldots$ be the canonical unit vectors of $B_K^{\mathbb{N}}$. Then $e_n \xrightarrow{\tau} 0$ and $p(e_n) = 1$ for every $n \in \mathbb{N}$. Hence, p is not τ-continuous. (See also Example 4.4.15 and Example 4.4.26.) □

4.4.50 Remark It is not true that τ^+ equals the p-topology. To show this, let $\lambda_0, \lambda_1, \lambda_2, \ldots \in B_K$ be such that $\frac{1}{\sqrt[n-1]{n-1}} < |\lambda_n| < \frac{1}{\sqrt[n]{n}} \quad (n \ge 4)$.
Let $x^n = \lambda_n e_n \quad (n \in \mathbb{N})$. Then $p(x^n) < \frac{1}{n} \quad (n \ge 4)$ and hence $p(x^n) \to 0$. As there does not exist a $\lambda \in B_K^-$ such that $x^n \in \lambda B_K^{\mathbb{N}}$ for large n we obtain from Proposition 4.4.48 that not $x^n \xrightarrow{\tau^+} 0$.

4.4.51 Remark Let p be as in he previous remark. In Example 4.4.15 we have seen that τ equals the p^--topology and hence, by Proposition 4.4.6 $\tau^- = \tau$.

We want to characterize the τ^+-open submodules of $B_K^{\mathbb{N}}$. But first we prove the following lemma.

4.4.52 Lemma *Let B be a submodule of $B_K^{\mathbb{N}}$. Let $c_0, c_1, c_2, \ldots > 0$ and $d_0, d_1, d_2, \ldots > 0$ such that*

$$\{x \in B_K^{\mathbb{N}} \mid |x_i| < c_i \text{ for all } i \in \mathbb{N}\} \subset B$$

and

$$\{x \in B_K^{\mathbb{N}} \mid |x_i| < d_i \text{ for all } i \in \mathbb{N}\} \subset B.$$

Then also

$$\{x \in B_K^{\mathbb{N}} \mid |x_i| < \max(c_i, d_i) \text{ for all } i \in \mathbb{N}\} \subset B.$$

Proof: Suppose $y \in \{x \in B_K^{\mathbb{N}} \mid |x_i| < \max(c_i, d_i) \text{ for all } i \in \mathbb{N}\}$. Let $S = \{j \in \mathbb{N} \mid |x_j| < c_j\}$. Let $u \in B_K^{\mathbb{N}}$ defined by

$$u_i = \begin{cases} y_i & \text{if } i \in S, \\ 0 & \text{if } i \in \mathbb{N} \setminus S, \end{cases}$$

and let $v \in B_K^{\mathbb{N}}$ be defined by

$$v_i = \begin{cases} 0 & \text{if } i \in S, \\ y_i & \text{if } i \in \mathbb{N} \setminus S. \end{cases}$$

Then

$$u \in \{x \in B_K^{\mathbb{N}} \mid |x_i| < c_i \text{ for all } i \in \mathbb{N}\} \subset B$$

and

$$v \in \{x \in B_K^{\mathbb{N}} \mid |x_i| < d_i \text{ for all } i \in \mathbb{N}\} \subset B.$$

Hence, $y = u + v \in B + B = B$. □

4.4.53 Proposition *Let U be a submodule of $B_K^{\mathbb{N}}$. Then*

$$U \in \tau^+ \iff \begin{array}{l} \text{There exist } \lambda_0, \lambda_1, \lambda_2, \ldots \in B_K \text{ with } |\lambda_0| \le |\lambda_1| \le |\lambda_2| \le \ldots < 1 \\ \text{and } \lim_{n\to\infty} |\lambda_n| = 1 \text{ such that} \\ \{x \in B_K^{\mathbb{N}} \mid |x_n| < |\lambda_n| \text{ for all } n \in \mathbb{N}\} \cap (B_K^{\mathbb{N}})^- \subset U. \end{array}$$

Proof: $\Rightarrow$) Let $\mu_0, \mu_1, \mu_2, \ldots \in B_K$ such that $|\mu_0| < |\mu_1| < |\mu_2| < \ldots$ and $\lim_{n\to\infty} |\mu_n| = 1$.
$\mu_0^{-1}U$ is τ-open and hence there exist $N_0 \in \mathbb{N}$ and an $\varepsilon_0 \in (0, 1]$ such that

$$\{x \in B_K^{\mathbb{N}} \mid |x_0| < \varepsilon_0, \ldots, |x_{N_0}| < \varepsilon_0\} \subset \mu_0^{-1}U.$$

Then the set

$$\{x \in B_K^{\mathbb{N}} \mid |x_0| < |\mu_0|\varepsilon_0, \ldots, |x_{N_0}| < |\mu_0|\varepsilon_0 \text{ and } |x_k| < |\mu_0| \text{ for all } k > N_0\}$$

is contained in U. Now $\mu_1^{-1}U$ is τ-open, hence there exist $N_1 \in \mathbb{N}$ and an $\varepsilon_1 \in (0, 1]$ such that

$$\{x \in B_K^{\mathbb{N}} \mid |x_0| < \varepsilon_1, \ldots, |x_{N_1}| < \varepsilon_1\} \subset \mu_1^{-1}U.$$

Then the set

$$\{x \in B_K^{\mathbb{N}} \mid |x_0| < |\mu_1|\varepsilon_1, \ldots |x_{N_1}| < |\mu_1|\varepsilon_1 \text{ and } |x_k| < |\mu_1| \text{ for all } k > N_1\}$$

is contained in U. We can arrange that $N_1 > N_0$. By Lemma 4.4.52 then also

$$\{x \in B_K^{\mathbb{N}} \mid |x_0| < |\mu_0|\varepsilon_0, \ldots |x_{N_0}| < |\mu_0|\varepsilon_0, |x_{N_0+1}| < |\mu_0|, \\ \ldots, |x_{N_1}| < |\mu_0|, |x_k| < |\mu_1| \text{ for all } k > N_1\} \subset U.$$

Repeating the process we obtain $N_0 < N_1 < N_2 < \ldots \in \mathbb{N}$ such that

$$\{x \in B_K^{\mathbb{N}} \mid |x_0| < |\mu_0|\varepsilon_0, \ldots, |x_{N_0}| < |\mu_0|\varepsilon_0, |x_{N_0+1}| < |\mu_0|, \ldots, |x_{N_1}| < |\mu_0|, \\ |x_{N_1+1}| < |\mu_1|, \ldots, |x_{N_2}| < |\mu_1|, \ldots, |x_{N_{j-1}+1}| < |\mu_{j-1}|, \ldots, \\ |x_{N_j}| < |\mu_{j-1}|, |x_k| < |\mu_j| \text{ for all } k > N_j\} \subset U$$

for every $j \in \mathbb{N}$. Let

$$\begin{aligned}V = \{x \in B_K^{\mathbb{N}} \mid |x_0| < |\mu_0|\varepsilon_0, \dots, |x_{N_0}| < |\mu_0|\varepsilon_0, |x_{N_0+1}| < |\mu_0|, \dots,\\ |x_{N_1}| < |\mu_0|, |x_{N_1+1}| < |\mu_1|, \dots, |x_{N_2}| < |\mu_1|, |x_{N_2+1}| < |\mu_2|,\\ \dots, |x_{N_3}| < |\mu_2|, |x_{N_3+1}| < |\mu_3|, \dots, |x_{N_4}| < |\mu_3|, \dots\}.\end{aligned}$$

We prove that $V \cap (B_K^{\mathbb{N}})^- \subset U$. To this end let $y \in V \cap (B_K^{\mathbb{N}})^-$. Then $y \in (B_K^{\mathbb{N}})^-$ and hence there exists a $\lambda \in B_K^-$ with $|\lambda| > |\mu_0|$ such that $y \in \lambda B_K^{\mathbb{N}}$. Let $j \in \mathbb{N}$ be such that $|\mu_j| > |\lambda| > |\mu_{j-1}|$.
Now $y \in V$ and hence

$$\begin{aligned}|y_i| &< |\mu_0|\varepsilon_0 \text{ for all } i \le N_0,\\ |y_i| &< |\mu_0| \text{ for all } N_0 < i \le N_1,\\ |y_i| &< |\mu_1| \text{ for all } N_1 < i \le N_2,\\ &\vdots\\ |y_i| &< |\mu_{j-1}| \text{ for all } N_{j-1} < i \le N_j\end{aligned}$$

and also $|y_i| < |\lambda|$ for all $i \in \mathbb{N}$ thus in particular $|y_i| < |\lambda| < |\mu_j|$ for all $i > N_j$. Hence,

$$\begin{aligned}y \in \{x \in B_K^{\mathbb{N}} \mid |x_0| < |\mu_0|\varepsilon_0, \dots, |x_{N_0}| < |\mu_0|\varepsilon_0, |x_{N_0+1}| < |\mu_0|, \dots,\\ |x_{N_1}| < |\mu_0|, |x_{N_1+1}| < |\mu_1|, \dots, |x_{N_2}| < |\mu_1|, \dots,\\ |x_{N_{j-1}+1}| < |\mu_{j-1}|, \dots, |x_{N_j}| < |\mu_{j-1}|, |x_k| < |\mu_j| \text{ for all } k > N_j\} \subset U.\end{aligned}$$

$\Leftarrow$) Let $\lambda_0, \lambda_1, \lambda_2, \dots \in B_K$ with $|\lambda_0| \le |\lambda_1| \le |\lambda_2| \le \dots$ and $\lim_{n\to\infty} |\lambda_n| = 1$ such that $\{x \in B_K^{\mathbb{N}} \mid |x_n| < |\lambda_n| \text{ for all } n \in \mathbb{N}\} \cap (B_K^{\mathbb{N}})^- \subset U$. Let $\mu \in K$ with $|\mu| > 1$. Let $N \in \mathbb{N}$ such that $|\lambda_n| > |\mu|^{-1}$ for all $n > N$. Then

$$\{x \in B_K^{\mathbb{N}} \mid |x_0| < |\mu||\lambda_0|, \dots, |x_N| < |\mu||\lambda_N|\} \subset \mu U$$

and hence μU is τ-open. We obtain that U is τ^+-open. □

4.4.54 Corollary τ^+ *is not metrizable.*

Proof: Suppose τ^+ is metrizable. By applying Proposition 4.4.32 we obtain that there exists a $\lambda \in B_K^-$ such that $\overline{\lambda B_K^{\mathbb{N}}}$ is τ^+-open. Now $\overline{\lambda B_K^{\mathbb{N}}} = \lambda B_K^{\mathbb{N}}$ and hence the latter set is τ^+-open. This is in contradiction with the previous proposition.
Hence τ^+ is not metrizable. □

In Proposition 4.4.48 we have seen that for a sequence $(x^n)_{n\in\mathbb{N}}$ in $B_K^{\mathbb{N}}$ with $x^n \xrightarrow{\tau^+} 0 \quad (n \to \infty)$ there exists a $\lambda \in B_K^-$ such that $x^n \in \lambda B_K^{\mathbb{N}}$ for large n. However, in the following example we construct a net $(x_\alpha)_{\alpha\in I}$ in $B_K^{\mathbb{N}}$ with $x_\alpha \xrightarrow{\tau^+} 0$ such that for every $\lambda \in B_K^-$ the set $\{\alpha \in I \mid x_\alpha \notin \lambda A\}$ is cofinal.

4.4.55 Example Let

$$V = \{(\mu_0, \mu_1, \mu_2, \dots) \in B_K^{\mathbb{N}} \mid |\mu_0| \le |\mu_1| \le |\mu_2| \le \dots < 1 \text{ and } \lim_{n\to\infty} |\mu_n| = 1\}.$$

Let $I = \{(n, v) \mid n \in \mathbb{N},\ v \in V\}$. Let the partial order $\succ$ on I be defined as follows. Let $u = (\mu_0, \mu_1, \mu_2, \dots) \in V$, $v = (\nu_0, \nu_1, \nu_2, \dots) \in V$ and $m, n \in \mathbb{N}$. Then

$$(m, u) \succ (n, v) \text{ if } |\mu_i| \le |\nu_i| \text{ for all } i \in \mathbb{N}.$$

Let $(x_\alpha)_{\alpha\in I}$ be the net in A defined by

$$x_{(n,v)} = (\nu_0, \dots, \nu_n, 0, 0, \dots), \text{ for } v = (\nu_0, \nu_1, \nu_2, \dots) \in V \text{ and } n \in \mathbb{N}.$$

It is not hard to see that for every $\lambda \in B_K^-$ there exist large α for which $x_\alpha \notin \lambda B_K^{\mathbb{N}}$. We prove that $x_\alpha \xrightarrow{\tau^+} 0$. We first observe that $x_\alpha \in (B_K^{\mathbb{N}})^-$ for all $\alpha \in I$. Let U be a τ^+ open submodule of A. By Proposition 4.4.53 there exist $\lambda_0, \lambda_1, \lambda_2, \dots \in B_K$ with $|\lambda_0| \le |\lambda_1| \le |\lambda_2| \le \dots < 1$ and $\lim_{n\to\infty} |\lambda_n| = 1$ such that $\{x \in B_K^{\mathbb{N}} \mid |x_n| \le |\lambda_n| \text{ for all } n \in \mathbb{N}\} \cap (B_K^{\mathbb{N}})^- \subset U$.
Let $y = (0, \lambda_0, \lambda_1, \lambda_2, \dots) \in I$. Let $\alpha \in I$ such that $\alpha \succ y$. Let $n \in \mathbb{N}$ and $u = (\mu_0, \mu_1, \mu_2, \dots) \in V$ such that $\alpha = (n, u)$. Then $|\mu_i| \le |\lambda_i|$ for all $i \in \mathbb{N}$ and hence

$$x_\alpha = (\mu_0, \dots, \mu_n, 0, 0, \dots) \in \{x \in B_K^{\mathbb{N}} \mid |x_n| \le |\lambda_n| \quad (n \in \mathbb{N})\} \cap (B_K^{\mathbb{N}})^- \subset U.$$

We see that $x_\alpha \xrightarrow{\tau^+} 0$.

4.5 Topologies on $\mathcal{L}(A, B)$

Introduction

4.5.1 Definition Let A and B be locally convex B_K-modules. By $\mathcal{L}(A, B)$ we denote the set of all continuous homomorphisms $A \to B$.

4.5.2 Proposition *Let A, B be locally convex B_K-modules. Then $\mathcal{L}(A, B)$ is a B_K-module with respect to the natural operations.*

The proof of this proposition is obvious. For locally convex K-vector spaces (E, τ) and (F, σ) it is customary to use the notation $\mathcal{L}(E, F)$ for the space of all continuous linear maps from (E, τ) to (F, σ). From Proposition 2.1.24 we know that each homomorphism $(E, \tau) \to (F, \sigma)$ is linear and hence the use of $\mathcal{L}$ in Definition 4.5.1 does not cause ambiguity.

4.5.3 Remark Let $(A, \|\ \|)$ and $(B, \|\ \|')$ be normed B_K-modules. Then the well-known formula

$$T \mapsto \sup_{\substack{x\in A\\ x\neq 0}} \frac{\|T(x)\|'}{\|x\|} \quad (T \in \mathcal{L}(A, B))$$

does, in general, not define a norm on $\mathcal{L}(A, B)$.
For example, let the valuation on K be non-trivial. Let ν on B_K be defined by $\nu(\lambda) = |\lambda|^2 \quad (\lambda \in B_K)$. From Proposition 3.3.15 we obtain that ν is a norm on B_K and $\nu \sim |\ |$. Hence, $T : (B_K, \nu) \to (B_K, |\ |)$ defined by $T(\lambda) = \lambda \quad (\lambda \in B_K)$ is a continuous homomorphism.
Now $\frac{|T(\lambda)|}{\nu(\lambda)} = \frac{1}{|\lambda|}$ for every $\lambda \in B_K \setminus \{0\}$ and hence

$$\sup_{\substack{\lambda \in B_K \\ \lambda \neq 0}} \frac{|T(\lambda)|}{\nu(\lambda)} = \infty.$$

For normed K-vector spaces $(E, \|\ \|)$ and $(F, \|\ \|')$ the above formula is a norm on $\mathcal{L}(A, B)$, which induces the topology of uniform convergence on bounded subsets of E
For normed B_K-modules $(A, \|\ \|)$ and $(B, \|\ \|')$ the topology on $\mathcal{L}(A, B)$ of uniform convergence on bounded subsets of A is, in general, not normable. For example, let $A = K^{\mathbb{N}}$, provided with the product topology. The product topology is metrizable and hence, by Theorem 3.4.15, also normable as a locally convex module topology. In vector space theory it is a well known fact that $\mathcal{L}(K^{\mathbb{N}}, K) = (K^{\mathbb{N}})' = K^{(\mathbb{N})}$, where $K^{(\mathbb{N})}$ is provided with the strongest locally convex vector space topology. Furthermore, $K^{(\mathbb{N})}$ is not metrizable and hence not normable as a B_K-module.

Topologies on $\mathcal{L}(A, B)$ defined by Bornologies

4.5.4 Definition Let (A, τ) be a locally convex B_K-module. A *bornology* on A is a non-empty collection $\mathfrak{S}$ of bounded subsets of A with the following property.

If $X, Y \in \mathfrak{S}$, then there exists a $Z \in \mathfrak{S}$ such that $X \cup Y \subset Z$.

4.5.5 Definition Let (A, τ) and (B, σ) be locally convex B_K-modules. Then the topology, on $\mathcal{L}(A, B)$, of the uniform convergence on elements of $\mathfrak{S}$ is defined as follows.
Let $(\varphi_\alpha)_{\alpha \in I}$ be a net in $\mathcal{L}(A, B)$ and let $\varphi \in \mathcal{L}(A, B)$. Then $\varphi_\alpha \to \varphi$ in the topology of the uniform convergence on elements of $\mathfrak{S}$ if $\varphi_\alpha(x) - \varphi(x) \xrightarrow{\sigma} 0$ uniformly on X, for every $X \in \mathfrak{S}$.

The B_K-module $\mathcal{L}(A, B)$ provided with the topology of the uniform convergence on elements of $\mathfrak{S}$ is denoted $\mathcal{L}_{\mathfrak{S}}(A, B)$.

4.5.6 Proposition *Let (A, τ) and (B, σ) be locally convex B_K-modules. Let $\mathfrak{S}$ be a bornology on A. Then the topology of the uniform convergence on elements of $\mathfrak{S}$ is a locally convex topology on $\mathcal{L}(A, B)$ and the collection*

$$C_{\mathfrak{S}} = (\{\varphi \in \mathcal{L}(A, B) \mid \varphi(X) \subset V\})_{X \in \mathfrak{S},\ V \text{ open submodule of } B}$$

is a base of zero neighbourhoods for this topology.

Proof: It is not hard to verify that $C_{\mathfrak{S}}$ is a collection of submodules of $\mathcal{L}(A,B)$. Let $\mathcal{F} = \{U_1,\dots,U_n\}$ be a finite subcollection of $C_{\mathfrak{S}}$. We prove that there exists a $V \in C_{\mathfrak{S}}$ such that $V \subset \bigcap \mathcal{F}$. Let $X_1,\dots,X_n \in \mathfrak{S}$ and let $V_1,\dots,V_n$ be open submodules of B such that

$$U_i = \{\varphi \in \mathcal{L}(A,B) \mid \varphi(X_i) \subset V_i\} \text{ for all } i \in \{1,\dots,n\}.$$

As $\mathfrak{S}$ is a bornology on A there exists a $Z \in \mathfrak{S}$ such that $X_1 \cup \dots \cup X_n \subset Z$. Let $W = V_1 \cap \dots \cap V_n$. Then W is an open submodule of B and hence $\{\varphi \in \mathcal{L}(A,B) \mid \varphi(Z) \subset W\} \in C_{\mathfrak{S}}$. Furthermore,

$$\{\varphi \in \mathcal{L}(A,B) \mid \varphi(Z) \subset W\} \subset \bigcap_{i=1}^{n} U_i.$$

Suppose $|K|$ is non-trivial. Let $U \in C_{\mathfrak{S}}$. We prove that U is absorbing. To this end let $X \in \mathfrak{S}$ and let V be an open submodule of B such that

$$U = \{\varphi \in \mathcal{L}(A,B) \mid \varphi(X) \subset V\}.$$

Let $\psi \in \mathcal{L}(A,B)$. Then $p_V \circ \psi$ is a continuous seminorm on A (recall from Proposition 3.3.30 that the seminorm p_V on B is defined by $p_V(x) = 0$ for $x \in V$, $p_V(x) = 1$ for $x \in B \setminus V$ and that p_V is continuous). As X is bounded we obtain from Theorem 4.3.5 that $\lim_{|\lambda|\to 0} p_V \circ \psi(\lambda x) = 0$ uniformly on X. Hence there exists a $\mu \in B_K \setminus \{0\}$ such that $p_V \circ \psi(\mu x) < 1$ for every $x \in A$. That is to say that $(\mu\psi)(X) = \psi(\mu X) \subset V$ and hence $\mu\psi \in U$.
From Proposition 3.1.26 we obtain that the $C_{\mathfrak{S}}$-topology is a locally convex topology and that $C_{\mathfrak{S}}$ is a base of zero neighbourhoods for the $C_{\mathfrak{S}}$-topology. It now suffices to prove that the $C_{\mathfrak{S}}$-topology and the topology of the uniform convergence on elements of $\mathfrak{S}$ coincide. An analogous assertion is known from vector space theory (see [6] and [16]); the proof is similar. □

4.5.7 Proposition *Let (A,τ) and (B,σ) be locally convex B_K-modules. Let $\mathfrak{S}$ be a bornology on A. Then $q_{X,p} : \mathcal{L}(A,B) \to [0,\infty)$ defined by*

$$q_{X,p}(\varphi) = \sup_{x\in X} p\,(\varphi(x)) \quad (\varphi \in \mathcal{L}(A,B))$$

is a seminorm on $\mathcal{L}(A,B)$ for every $X \in \mathfrak{S}$ and every bounded continuous seminorm p on B. Moreover, the collection

$$(q_{X,p})_{X\in\mathfrak{S},\ p \text{ bounded continuous seminorm on } B}$$

is a weak base of seminorms for the topology of the uniform convergence on elements of $\mathfrak{S}$.

Proof: Let $X \in \mathfrak{S}$ and p a bounded continuous seminorm on B.
1. We prove that $q_{X,p}$ is a seminorm on $\mathcal{L}(A,B)$. Properties (i), (ii) and (iii) are easy to verify. We check property (iv).
If $|K|$ is trivial there is nothing to check, so suppose $|K|$ is non-trivial. Let $\varphi \in \mathcal{L}(A,B)$ and let $\lambda_1,\lambda_2,\lambda_3,\dots \in B_K$ such that $\lim_{n\to\infty} \lambda_n = 0$. As X is

bounded and φ is continuous the image $\varphi(X)$ is bounded. By using Theorem 4.3.5 we obtain that $\lim_{\lambda\to 0} p(\lambda y) = 0$ uniformly on $\varphi(X)$. Then $q_{X,p}(\lambda_n\varphi) = \sup_{x\in X} p(\lambda_n\varphi(x)) = \sup_{y\in\varphi(X)} p(\lambda_n y) \to 0 \quad (n\to\infty)$.
2. We next prove that $q_{X,p}$ is continuous. To this end let $\varepsilon > 0$. Then the submodule $\mathrm{B}_p(\frac{1}{2}\varepsilon) := \{y \in B \mid p(y) < \frac{1}{2}\varepsilon\}$ is open in B and

$$\{\varphi \in \mathcal{L}(A,B) \mid \varphi(X) \subset \mathrm{B}_p(\frac{1}{2}\varepsilon)\} \subset \{\varphi \in \mathcal{L}(A,B) \mid q_{X,p}(\varphi) < \varepsilon\}.$$

Thus, $\{\varphi \in \mathcal{L}(A,B) \mid q_{X,p}(\varphi) < \varepsilon\}$ is open in $\mathcal{L}_{\mathfrak{S}}(A,B)$. From Proposition 3.3.7 we obtain that $q_{X,p}$ is a continuous seminorm on $\mathcal{L}_{\mathfrak{S}}(A,B)$.
3. Finally we prove that the collection

$$(q_{X,p})_{X\in\mathfrak{S},\ p \text{ bounded continuous seminorm on } B}$$

is a weak base of seminorms for the topology of uniform convergence on elements of $\mathfrak{S}$. To this end, let U be an open submodule of $\mathcal{L}_{\mathfrak{S}}(A,B)$. Then there exists an $X \in \mathfrak{S}$ and an open submodule V of B such that

$$\{\varphi \in \mathcal{L}(A,B) \mid \varphi(X) \subset V\} \subset U.$$

Now p_V is a continuous seminorm on B and

$$\{\varphi \in \mathcal{L}(A,B) \mid q_{X,p_V}(\varphi) < 1\} \subset \{\varphi \in \mathcal{L}(A,B) \mid \varphi(X) \subset V\} \subset U.$$

□

4.5.8 Examples Let A and B be locally convex B_K-modules. We have the following natural locally convex topologies on $\mathcal{L}(A,B)$ defined by a bornology.

1. The topology of uniform convergence on all bounded sets. This topology is obtained by taking for $\mathfrak{S}$ the collection of all bounded subsets on A.

2. The topology of pointwise convergence. This topology is obtained by taking for $\mathfrak{S}$ the collection of all finite subsets of A. It is not hard to see that the collection $(q_{x,p})_{x\in A,\ p \text{ continuous seminorm on } B}$, where $q_{x,p}$ is defined by $q_{x,p}(\varphi) = p(\varphi(x)) \quad (\varphi \in \mathcal{L}(A,B))$, is a generating collection of seminorms for the topology.

3. The topology of uniform convergence on 'bounded submodules of finite rank'. This is obtained by taking for $\mathfrak{S}$ the collection of all submodules of A that are a member of $\mathcal{B}_K$. (Observe that by Proposition 4.3.12 these submodules are bounded.)

4.5.9 Proposition *Let the valuation on K be dense. For locally convex B_K-modules (A,τ) and (B,ν) let, on $\mathcal{L}(A,B)$, σ_{fr} be the topology of the uniform convergence on bounded submodules of finite rank and let σ be the topology of pointwise convergence. Then $\sigma_{\mathrm{fr}}^{-} \le \sigma \le \sigma_{\mathrm{fr}}$.*

Proof: 1. We prove that $\sigma_{\mathrm{fr}}^{-} \le \sigma$. Let $(\varphi_\alpha)_{\alpha\in I}$ be a net in $\mathcal{L}(A,B)$ such that $\varphi_\alpha \xrightarrow{\sigma} 0$. Let $\lambda \in B_K^-$. Let U be a σ_{fr}-open submodule of $\mathcal{L}(A,B)$. Then there exists an $X \subset A$ with $X \in \mathcal{B}_K$ and an open submodule V of B such that $\{\varphi \in \mathcal{L}(A,B) \mid \varphi(X) \subset V\} \subset U$. There exist $x_1,\dots,x_n \in A$ such that $\lambda X \subset \mathrm{co}\{x_1,\dots,x_n\} \subset X$. Now $\{\varphi \in \mathcal{L}(A,B) \mid \varphi(\{x_1,\dots,x_n\}) \subset V\}$ is σ-open and as V is a submodule we obtain that

$$\{\varphi \in \mathcal{L}(A,B) \mid \varphi(\{x_1,\dots,x_n\}) \subset V\} = \{\varphi \in \mathcal{L}(A,B) \mid \varphi(\mathrm{co}\{x_1,\dots,x_n\}) \subset V\}$$

and hence the latter set is also σ-open. Thus,

$$\varphi_\alpha \in \{\varphi \in \mathcal{L}(A,B) \mid \varphi(\mathrm{co}\{x_1,\dots,x_n\}) \subset V\}$$

for large α. Let $y \in X$. Then $\lambda y \in \mathrm{co}\{x_1,\dots,x_n\}$ which implies that $\lambda\varphi_\alpha(y) = \varphi_\alpha(\lambda y) \in \varphi_\alpha(\mathrm{co}\{x_1,\dots,x_n\}) \subset V$ for large α. We obtain that $\lambda\varphi_\alpha \in \{\varphi \in \mathcal{L}(A,B) \mid \varphi(X) \subset V\}$ for large α.
We see that $\lambda\varphi_\alpha \xrightarrow{\sigma_{\mathrm{fr}}} 0$ for every $\lambda \in B_K^-$. From Proposition 4.4.10 it follows that $\varphi_\alpha \xrightarrow{\sigma_{\mathrm{fr}}^-} 0$.
2. It is obvious that $\sigma \le \sigma_{\mathrm{fr}}$. □

In the following example we see that the inequalities in the previous proposition are in general strict.

4.5.10 Example Let the valuation on K be dense. Let $A = (B_K, |\ |)$ and let $B = (B_K/B_K^-, d)$, where d is the discrete topology on B. Then d is the quotient topology of the $|\ |$-topology. Let $\pi : A \to B$ be the quotient map. Then $\pi \in \mathcal{L}(A,B)$. Consider the sequence $\pi,\pi,\pi,\dots$. Then $\lambda\pi = 0$ and hence $\lambda\pi \xrightarrow{\sigma_{\mathrm{fr}}} 0$ for every $\lambda \in B_K^-$. This implies that $\pi \xrightarrow{\sigma_{\mathrm{fr}}^-} 0$. But not $\pi \xrightarrow{\sigma} 0$ since $\pi(1) = 1 + B_K^- \ne 0$. Hence, $\sigma_{\mathrm{fr}}^- < \sigma$.
Let $A = (B_K^-, |\ |)$. Let $\lambda_1,\lambda_2,\lambda_3,\dots \in B_K^-$ such that $0 < |\lambda_1| < |\lambda_2| < \dots$ and $\lim_{n\to\infty} |\lambda_n| = 1$. Let $B = B_K^-/\lambda_1 B_K^-$ equipped with the discrete topology, which is the quotient topology of $|\ |$. Let $f_n : A \to B$ be defined by

$$f_n(\mu) = \frac{\lambda_1}{\lambda_n}(\mu + \lambda_1 B_K^-) \quad (\mu \in A) \quad (n \ge 1).$$

Then $f_n \in \mathcal{L}(A,B)$ for every $n \ge 1$, for f_1 is the quotient map and $f_n = \frac{\lambda_1}{\lambda_n} f_1$, where $\frac{\lambda_1}{\lambda_n} \in B_K$. Now $f_n \xrightarrow{\sigma} 0$. For let $\mu \in A$. There exists an $n \in \mathbb{N}$ such that $|\mu| < |\lambda_n|$. For $m \ge n$ we have that $\lambda_m^{-1}\mu \in B_K^-$ and hence

$$f_m(\mu) = \frac{\lambda_1}{\lambda_m}(\mu + \lambda_1 B_K^-) = \lambda_1(\lambda_m^{-1}\mu) + \lambda_1 B_K^- = \lambda_1 B_K^- = 0.$$

Hence, $f_n(\mu) \to 0$ $(n \to \infty)$ in B. Should $f_n \xrightarrow{\sigma_{\mathrm{fr}}} 0$, then $f_n \to 0$ uniformly on B_K^- since $B_K^- \in \mathcal{B}_K$. But $f_n(\lambda_n) = \lambda_1 + \lambda_1 B_K^-$ for every $n \ge 1$. Hence, $\sigma < \sigma_{\mathrm{fr}}$.

4.5.11 Remark If $|K|$ is discrete then, by Proposition 2.2.38, every bounded submodule of finite rank is finitely generated. It is not hard to prove that then $\sigma_{\mathrm{fr}} = \sigma$.

Duals

For every non-zero K-vector space E we have that the algebraic dual E^* (i.e. the set of all linear maps $E \to K$) has non-trivial elements. That is to say that $E^* \neq \{0\}$. For a non-zero B_K-module A, however, it may happen that $\operatorname{Hom}(A, K) = \{0\}$, for instance if A is a torsion module. Thus, in order to define a suitable notion of an algebraic dual for B_K-modules we want the role of K in the above be taken over by a B_K-module C of rank 1 such that $\operatorname{Hom}(A, C) \neq \{0\}$ for every B_K-module A. If the valuation on K is trivial then $K(= B_K)$ is the only B_K-module of rank 1. Furthermore, every B_K-module A is a K-vector space and $\operatorname{Hom}(A, K) = L(A, K) \neq \emptyset$. Hence we can take $C = K$. In the next proposition we will see that, if $|K|$ is non-trivial, we also have only one choice for C.

4.5.12 Proposition *Let the valuation on K be non-trivial. Let C be a B_K-module of rank 1 such that* $\operatorname{Hom}(A, C) \neq \{0\}$ *for every non-zero B_K-module A. Then $C \sim K/B_K^-$.*

Proof: 1. As $\operatorname{rank} C = 1$ there exists an absolutely convex subset B of K and a surjective homomorphism $\varphi : B \to C$. Let $S = \operatorname{Ker} \varphi$. Then $C \sim B/S$.
2. We prove $B = K$. By assumption there exists a homomorphism $\rho : K \to C$ with $\rho \neq 0$. Let $\lambda_0 \in K$ be such that $\rho(\lambda_0) \neq 0$. Let $\mu_0 \in B$ be such that $\varphi(\mu_0) = \rho(\lambda_0)$. Then $\mu_0 \notin S$ and hence $S \subset \mathrm{B}(0, |\mu_0|^-)$.
Let $\nu \in K$. We prove that $\nu \in B$. If $|\nu| \leq |\mu_0|$ we are done, so suppose $|\nu| > |\mu_0|$. Then $\frac{\nu}{\mu_0}\lambda_0 \in K$ and $\frac{\mu_0}{\nu}\rho(\frac{\nu}{\mu_0}\lambda_0) = \rho(\lambda_0) \neq 0$. This implies that also $\rho(\frac{\nu}{\mu_0}\lambda_0) \neq 0$. Let $\mu \in B$ such that $\varphi(\mu) = \rho(\frac{\nu}{\mu_0}\lambda_0)$. Then $\varphi(\frac{\mu_0}{\nu}\mu) = \rho(\lambda_0)$ and hence $\frac{\mu_0}{\nu}\mu - \mu_0 \in S$. Then $|\frac{\mu_0}{\nu}\mu - \mu_0| < |\mu_0|$ and hence $|\frac{\mu_0}{\nu}\mu| = |\mu_0|$, which implies that $|\nu| = |\mu|$. As $\mu \in B$ we obtain that also $\nu \in B$.
Hence, $B = K$. Then $C \sim K/S$.
3. Next we prove that S has the form $\mathrm{B}(0, |\mu|^-)$ for some $\mu \in K \setminus \{0\}$. By assumption there exists a homomorphism $\rho : B_K/B_K^- \to C$ with $\rho \neq 0$. Then $\rho(1 + B_K^-) \neq 0$. Let $\mu \in B$ be such that $\varphi(\mu) = \rho(1 + B_K^-)$. Then for every $\lambda \in \mathrm{B}(0, |\mu|^-)$ we have that

$$\varphi(\lambda) = \frac{\lambda}{\mu}\varphi(\mu) = \frac{\lambda}{\mu}\rho(1 + B_K^-) = \rho(\frac{\lambda}{\mu} + B_K^-) = \rho(0) = 0.$$

We see that $\mathrm{B}(0, |\mu|^-) \subset S$. As $\mu \notin S$ we obtain that $S = \mathrm{B}(0, |\mu|^-)$.
4. The map $\psi : K/B_K^- \to K/S$ defined by $\psi(\lambda + B_K^-) = \mu\lambda + S$ is a bijective homomorphism and hence $C \sim K/B_K^-$. □

We denote K/B_K^- by K_1 ($K_1 = K$ for trivially valued K). For a B_K-module A we call $\operatorname{Hom}(A, K_1)$ the *algebraic dual* of A. In the following theorem we see that this causes no ambiguity in the case that A is a K-vector space.

4.5.13 Theorem *Let E be a K-vector space. Then* $\operatorname{Hom}(E, K_1) \sim E^*$.

Proof: If $|K|$ is trivial there is nothing to prove, so suppose $|K|$ is non-trivial.
1. Let $\pi : K \to K_1$ be the quotient map. We define the homomorphism

$T : E^* \to \mathrm{Hom}(E, K_1)$ by $T(f) = \pi \circ f \quad (f \in E^*)$.
Let $f \in E^*$ such that $Tf = 0$. Then $\pi \circ f = 0$ and hence $f(E) \subset B_K^-$. As f is linear this implies $f = 0$. We see that T is injective.
2. We now prove that T is surjective.
Let $\varphi \in \mathrm{Hom}(E, K_1)$. We construct an $f \in E^*$ such that $Tf = \varphi$, in other words such that the following diagram commutes.

$$\begin{tikzcd} E \arrow[r, dashed, "f"] \arrow[dr, "\varphi"'] & K \arrow[d, "\pi"] \\ & K_1 \end{tikzcd}$$

Let $x \in E$. For every $\lambda \in K$ it is required that $f(\lambda x) \in \pi^{-1}(\varphi(\lambda x))$, hence, by linearity of f,

$$f(x) \in \bigcap_{\lambda \neq 0} \lambda^{-1}\pi^{-1}(\varphi(\lambda x)).$$

We shall prove that this intersection is a singleton. In fact, let $\lambda, \mu \in K$ with $|\lambda| \geq |\mu| > 0$. Let $y \in \lambda^{-1}\pi^{-1}(\varphi(\lambda x))$. Then there exists a $z \in \pi^{-1}(\varphi(\lambda x))$ such that $y = \lambda^{-1}z$. Now $\mu\lambda^{-1} \in B_K$ and

$$\pi(\mu\lambda^{-1}z) = (\mu\lambda^{-1})\pi(z) = (\mu\lambda^{-1})\varphi(\lambda x) = \varphi((\mu\lambda^{-1})\lambda x) = \varphi(\mu x).$$

Thus, $\mu\lambda^{-1}z \in \pi^{-1}(\varphi(\mu x))$. Then $y = \lambda^{-1}z = \mu^{-1}(\mu\lambda^{-1})z$ and $\mu^{-1}(\mu\lambda^{-1})z \in \mu^{-1}\pi^{-1}(\varphi(\mu x))$. We obtain that

$$\lambda^{-1}\pi^{-1}(\varphi(\lambda x)) \subset \mu^{-1}\pi^{-1}(\varphi(\mu x)).$$

Let $\lambda \in K \setminus \{0\}$. Let $v, w \in \lambda^{-1}\pi^{-1}(\varphi(\lambda x))$. Then $\lambda v, \lambda w \in \pi^{-1}(\varphi(\lambda x))$. Thus,

$$\pi(\lambda(v - w)) = \pi(\lambda v) - \pi(\lambda w) = \varphi(\lambda x) - \varphi(\lambda x) = 0$$

and hence $\lambda(v - w) \in B_K^-$. Then $|v - w| < |\lambda^{-1}|$.
We obtain that $\operatorname{diam} \lambda^{-1}\pi^{-1}(\varphi(\lambda x)) \leq |\lambda|^{-1}$.
Summarizing we have the following. If $\lambda, \mu \in K$ are such that $|\lambda| \geq |\mu| > 0$ then $\lambda^{-1}\pi^{-1}(\varphi(\lambda x)) \subset \mu^{-1}\pi^{-1}(\varphi(\mu x))$ and

$$\lim_{|\lambda| \to \infty} \operatorname{diam} \lambda^{-1}\pi^{-1}(\varphi(\lambda x)) = 0.$$

As K is complete it follows that $\bigcap_{\lambda \neq 0} \lambda^{-1}\pi^{-1}(\varphi(\lambda x)) \neq \varnothing$. We define $f(x)$ such that $\bigcap_{\lambda \neq 0} \lambda^{-1}\pi^{-1}(\varphi(\lambda x)) = \{f(x)\}$.
In this way we find a map $f : E \to K$.
That $T \circ f = \varphi$ can be seen as follows. Let $x \in E$. Then

$$f(x) \in \bigcap_{\lambda \neq 0} \lambda^{-1}\pi^{-1}(\varphi(\lambda x)) \subset \pi^{-1}(\varphi(x))$$

and hence $T \circ f(x) = (\pi \circ f)(x) = \pi(f(x)) = \varphi(x)$.
We prove that f is linear.
Let $x, y \in E$. Let $\lambda \in K \setminus \{0\}$. Then

$$f(x) + f(y) \in \lambda^{-1}\pi^{-1}(\varphi(\lambda x)) + \lambda^{-1}\pi^{-1}(\varphi(\lambda y))$$

and

$$\lambda^{-1}\pi^{-1}(\varphi(\lambda x)) + \lambda^{-1}\pi^{-1}(\varphi(\lambda y)) = \lambda^{-1}(\pi^{-1}(\varphi(\lambda x)) + \pi^{-1}(\varphi(\lambda y))) =$$
$$\lambda^{-1}\pi^{-1}(\varphi(\lambda x) + \varphi(\lambda y)) = \lambda^{-1}\pi^{-1}(\varphi(\lambda(x+y))).$$

We see that $f(x)+f(y) \in \bigcap_{\lambda\neq 0} \lambda^{-1}\pi^{-1}(\varphi(\lambda(x+y))) = \{f(x+y)\}$. Thus, $f(x+y) = f(x) + f(y)$.
Let $x \in E$ and $\mu \in K$. If $\mu = 0$ then $f(\mu x) = f(0) = 0 = \mu f(x)$, so suppose $\mu \neq 0$. Let $\lambda \in K \setminus \{0\}$. Then $f(x) \in \mu^{-1}\lambda^{-1}\pi^{-1}(\varphi(\lambda\mu x))$ and hence $\mu f(x) \in \mu(\mu^{-1}\lambda^{-1})\pi^{-1}(\varphi(\lambda\mu x)) = \lambda^{-1}\pi^{-1}(\varphi(\lambda(\mu x)))$.
We see that $\mu f(x) \in \bigcap_{\lambda\neq 0} \lambda^{-1}\pi^{-1}(\varphi(\lambda(\mu x))) = \{f(\mu x)\}$.
Thus, $f(\mu x) = \mu f(x)$.
We obtain that $f \in E^*$ and $T \circ f = \varphi$.
We see that $T : E^* \to \operatorname{Hom}(E, K_1)$ is a bijective homomorphism. □

We now move to the topological dual of a locally convex B_K-module. From Proposition 4.1.5 we obtain that the discrete topology is the only locally convex Hausdorff topology on K_1 and from now on K_1 will be equipped with this topology.

4.5.14 Theorem *Let E be a locally convex K-vector space. Then E' and $\mathcal{L}(E, K_1)$ are isomorphic as B_K-modules.*

Proof: Again, if $|K|$ is trivial there is nothing to prove, so suppose $|K|$ is non-trivial. Let T be defined as in the proof of the previous proposition. We prove that $T(E') = \mathcal{L}(E, K_1)$.
Let $f \in E'$. Then $Tf = \pi \circ f$ is continuous as π is continuous and f is continuous, hence $Tf \in \mathcal{L}(E, K_1)$.
Let $\varphi \in \mathcal{L}(E, K_1)$. Then there exists an $f \in E^*$ such that $Tf = \varphi$. Then $\pi \circ f$ is continuous and hence $f^{-1}(B_K^-) = (\pi \circ f)^{-1}\{0\}$ is open in E.
Let $\lambda \in B_K$. Then $f^{-1}(\lambda B_K^-) = \lambda f^{-1}(B_K^-)$ and since E is a locally convex K-vector space we obtain that the latter set is open in E.
As $(\lambda B_K^-)_{\lambda\in B_K}$ is a base of zero neighbourhoods in K we obtain that f is continuous. Hence, $\varphi \in T(E')$.
As T is bijective it follows that $E' \sim \mathcal{L}(E, K_1)$. □

4.5.15 Remark In the proof we saw that for an $f \in E^*$ the fact that $\pi \circ f$ is continuous implies that $f \in E'$. If E is only equipped with a locally convex module topology and $f : E \to K$ such that $\pi \circ f$ is continuous, then f is not necessarily continuous.
For example, let the valuation on K be non-trivial. Let $E = K^{(\mathbb{N})}$. Let $\| \ \|$ on E be defined by

$$\|(\lambda_0, \lambda_1, \lambda_2, \dots)\| = (|\lambda_0| \wedge 1) \vee \max_{n\geq 1}(|\lambda_n|^n \wedge 1) \quad (\lambda_1, \lambda_2, \lambda_3, \dots \in K).$$

It is not hard to verify that $\| \ \|$ is a (module) norm on E.
Let $f : E \to K$ be defined by

$$f((\lambda_0, \lambda_1, \lambda_2, \dots)) = \sum_{n=0}^{\infty} \lambda_n \quad (\lambda_0, \lambda_1, \lambda_2, \dots \in K).$$

The map f is well defined since $\lambda_n = 0$ for large n. It is easy to see that f is linear.
Now f is not continuous. In fact, let $\lambda \in B_K$ with $0 < |\lambda| \le \frac{1}{2}$. Then $\|\lambda e_n\| = |\lambda|^n \quad (n \ge 1)$ and hence $\lim_{n\to\infty} \|\lambda e_n\| = 0$. (Here $e_0, e_1, e_2, \ldots$ are the canonical unit vectors of E.)
Now $f(\lambda e_n) = \lambda$ for every $n \ge 1$ and hence not $f(\lambda e_n) \to 0$.
The following shows that $\pi \circ f$ is continuous. Let $(x^n)_{n\in\mathbb{N}}$ be a sequence in E such that $\lim_{n\to\infty} \|x^n\| = 0$. For every $n \in \mathbb{N}$ let $\lambda_0^n, \lambda_1^n, \lambda_2^n, \ldots \in K$ such that $x^n = (\lambda_0^n, \lambda_1^n, \lambda_2^n, \ldots)$. Then $(|\lambda_0| \wedge 1) \vee \max_{i\ge 1}(|\lambda_i^n|^i \wedge 1) = \|x^n\| < \frac{1}{2}$ for large n. Hence, $|\lambda_i^n| < 1 \quad (i \in \mathbb{N})$ for large n. Then $|f(x^n)| = |\sum_{i=0}^{\infty} \lambda_i^n| \le \max_{i\in\mathbb{N}} |\lambda_i^n| < 1$ for large n (observe that the sum is actually finite).
Thus, $f(x^n) \in B_K^-$ for large n and hence $\pi \circ f(x^n) = 0$ for large n. Thus $\pi \circ f(x^n) \to 0$.

4.5.16 Notation Let A be a locally convex B_K-module. We will denote $\mathrm{Hom}\,(A, K_1)$ by A^{d} and $\mathcal{L}(A, K_1)$ by A'.

From the previous proposition we obtain that this last notation causes no ambiguity if A is a locally convex K-vector space. Note that in algebra, for an R-module A, the notation A^* is used to denote the set $\mathrm{Hom}\,(A, R)$.

We now prove the Hahn-Banach Theorem for B_K-modules (compare [31]). First an algebraic version, next a topological one.

4.5.17 Theorem *Let K be spherically complete. Let A be a B_K-module and let B be a submodule of A. Let $f : B \to K_1$ be a homomorphism. Then there exists a homomorphism $\tilde{f} : A \to K_1$ that is an extension of f.*

Proof: By a standard application of Zorn's lemma (compare the proof of Theorem 3.2.5) it suffices to prove the following. Let $x \in A \setminus B$. Then f extends to $B + \mathrm{co}\,\{x\}$.
Suppose B does not absorb x. Then the sum $B + \mathrm{co}\,\{x\}$ is direct. It is not hard to verify that $h : B \oplus \mathrm{co}\,\{x\}$ defined by $h(y + \lambda x) = f(y) \quad (y \in B,\ \lambda \in B_K)$ is a homomorphism that is an extension of f.
Suppose that B absorbs x. Let $V = \{\lambda \in B_K \mid \lambda x \in B\}$. Let $\mu \in V$. Then $\mu^{-1}\{f(\mu x)\} \ne \emptyset$ for every $\mu \in V$ and $\mu^{-1}\{f(\mu x)\} \subset \lambda^{-1}\{f(\lambda x)\}$ for every $\lambda, \mu \in V$ with $|\lambda| \le |\mu|$.
Hence the collection $(\mu^{-1}\{f(\mu x)\})_{\mu\in V}$ has the finite intersection property. As K_1 is linearly compact we obtain that $C := \bigcap_{\mu\in V} \mu^{-1}\{f(\mu x)\} \ne \emptyset$. Let $w \in C$. Then $\mu w = f(\mu x)$ for every $\mu \in V$.
We define $h : B + \mathrm{co}\,\{x\}$ by

$$h(y + \lambda x) = f(y) + \lambda w \quad (y \in B,\ \lambda \in B_K).$$

This is a good definition for suppose $y, z \in B$ and $\lambda, \mu \in B_K$ such that $y + \lambda x = z + \mu x$.
If $\lambda \in V$ then $\mu \in V$ and hence $h(z + \mu x) = f(z) + \mu w = f(z) + f(\mu x) = f(z + \mu x) = f(y + \lambda x) = f(y) + f(\lambda x) = f(y) + \lambda w = h(y + \lambda x)$.
If $\lambda \notin V$ then $\mu \notin V$ and $(\lambda - \mu)x = z - y \in B$. Then $h(z + \mu x) - h(y + \lambda x) =$

$(f(z)+\mu w)-(f(y)+\lambda w) = f(z-y)+(\lambda-\mu)w = f(z-y)-f((\lambda-\mu)x) = f((z-y)-(\lambda-\mu)x) = f(0) = 0$ and hence $h(z+\mu x) = h(y+\lambda x)$.
It is not hard to verify that h is a homomorphism and that h is an extension of f. □

4.5.18 Theorem *Let K be spherically complete. Let (A,τ) be a locally convex B_K-module and let B be a submodule of A. Let $f : B \to K_1$ be a continuous homomorphism. Then there exists an $\tilde{f} \in \mathcal{L}(A,K_1)$ extending f.*

Proof: As f is continuous we obtain that $\operatorname{Ker} f$ is open in B. Hence there exists an open submodule S of A such that $\operatorname{Ker} f = S \cap B$ (see Proposition 3.1.30). Let $\pi_1 : B \to B/\operatorname{Ker} f$ and $\pi_2 : A \to A/S$ be the quotient maps. Let $i : B \to A$ be the inclusion map. Then there exists an injective homomorphism $j : B/\operatorname{Ker} f \to A/S$ such that the following diagram commutes.

$$\begin{tikzcd} B \arrow[r, "i"] \arrow[d, "\pi_1"'] & A \arrow[d, "\pi_2"] \\ B/\operatorname{Ker} f \arrow[r, "j"'] & A/S \end{tikzcd}$$

From Theorem 2.1.25 we obtain that there exists an (injective) homomorphism $f_1 : B/\operatorname{Ker} f \to K_1$ such that $f_1 \circ \pi_1 = f$. By Theorem 4.5.17 there exists a homomorphism $\tilde{f}_1 : A/S \to K_1$ such that the following diagram commutes.

$$\begin{tikzcd} B \arrow[rr, "i"] \arrow[dr, "\pi_1"] \arrow[ddr, "f"'] & & A \arrow[dr, "\pi_2"] & \\ & B/\operatorname{Ker} f \arrow[rr, "j"] \arrow[d, "f_1"'] & & A/S \arrow[dll, dashed, "\tilde{f}_1"] \\ & K_1 & & \end{tikzcd}$$

Let $\tilde{f} = \tilde{f}_1 \circ \pi_2$. Then $\tilde{f}$ is an extension of f and $\tilde{f}$ is continuous, since $S \subset \operatorname{Ker} f$ and hence $\operatorname{Ker} f$ is open. □

4.5.19 Proposition *Let S be a B_K-module of rank 1 and $s \in S$, $s \neq 0$. Then there exists an $f \in \operatorname{Hom}(S,K_1)$ with $f(s) \neq 0$.*

Proof: Let C be an absolutely convex subset of K and let $\rho : C \to S$ be a surjective homomorphism. We may suppose that $B_K \subset C$ and $\rho(1) = s$. Then $\operatorname{Ker}\rho \subset B_K^-$. We define $\varphi : S \to K_1$ as follows. Let $x \in S$. Let $\lambda \in C$ such that $\rho(\lambda) = x$. Then $\varphi(x) := \lambda + B_K^-$.
Then $\varphi(x)$ is well defined for $\operatorname{Ker}\rho \subset B_K^-$. Furthermore, it is not hard to verify that φ is a homomorphism and from $\rho(1) = s$ we obtain that $\varphi(s) = 1 + B_K^- \neq 0$. □

4.5.20 Corollary *Let K be spherically complete. Let A be a B_K-module. Then A^d separates the points of A.*

Proof: Let $x \in A$, $x \neq 0$. Then $\operatorname{rank} \operatorname{co}\{x\} = 1$ and hence there exists a $\varphi : \operatorname{co}\{x\} \to K_1$ with $\varphi(x) \neq 0$. By using Theorem 4.5.17 we obtain that φ can be extended to a homomorphism $\hat{\varphi} : A \to K_1$. Then $\hat{\varphi} \in A^d$ and $\hat{\varphi}(x) \neq 0$. □

4.5.21 Theorem *Let K be spherically complete. Let (A, τ) be a locally convex Hausdorff B_K-module. Then A' separates the points of A.*

Proof: Let C be the collection of all open submodules of A. From Proposition 3.1.31 we obtain that the homomorphism φ defined by

$$\Phi(x)(U) = x + U \quad (U \in C,\ x \in A)$$

is a homeomorphism from A in $\prod_{U \in C} A/U$, For every $V \in C$ let P_V denote the projection map from $\prod_{U \in C} A/U$ on A/V.
Let $x \in A, x \neq 0$. As (A, τ) is Hausdorff there exists a $V \in C$ such that $x \notin V$. From the previous proposition we obtain that there exists a $\rho \in (A/V)^*$ with $\rho(x + V) \neq 0$. Then ρ is continuous as A/V is provided with the discrete topology. Now $\rho \circ P_V \circ \varphi : A \to K_1$ is a continuous homomorphism and $\rho \circ P_V \circ \varphi(x) \neq 0$. □

In the following example we will show that B_K and K_1 are duals of each other. If the valuation on K is trivial this means $K' = K$, which is a well-known fact. In the example we therefore assume that $|K|$ is non-trivial.

4.5.22 Example We first determine the dual of K_1. We have $K_1' = K_1^d$ for K_1 is provided with the discrete topology.
Let $\nu \in B_K$. Let $\varphi_\nu : K_1 \to K_1$ be defined by $\varphi_\nu(\lambda + B_K^-) = \nu\lambda + B_K^-$. Then $\varphi_\nu \in K_1'$. Let $T : B_K \to K_1'$ be defined by $T(\nu) = \varphi_\nu \quad (\nu \in B_K)$. It is not hard to see that T is a homomorphism.
Let $\nu \in B_K$ such that $\varphi_\nu = 0$. Then $\nu\lambda \in B_K^-$ for every $\lambda \in K$. This implies $\nu = 0$. Hence, T is injective. That T is also surjective we see as follows.
Let $\rho \in K_1'$. If $\rho = 0$ then $T(0) = \rho$. Suppose $\rho \neq 0$. Let $\lambda_0 \in K$ such that $\rho(\lambda_0 + B_K^-) = 1 + B_K^-$. Then $|\lambda_0| \geq 1$.
For every $\lambda \in K$ with $|\lambda| \geq |\lambda_0|$ let $V_\lambda \subset K$ be defined by

$$V_\lambda = \lambda^{-1}\pi^{-1}(\rho(\lambda + B_K^-)).$$

(Here $\pi : K \to K_1$ is the quotient map.) Then each V_λ is a convex set and $\operatorname{diam} V_\lambda = s|\lambda|^{-1}$, where $s = \sup |B_K^-|$. Furthermore, if $\lambda, \mu \in K$ such that $|\lambda| \geq |\mu| \geq |\lambda_0|$ then $V_\lambda \subset V_\mu$.
As K is complete we obtain that there exists a $\nu \in K$ with $\bigcap_{|\lambda| \geq |\lambda_0|} V_\lambda = \{\nu\}$. Then $\nu \in V_{\lambda_0}$, hence there exists a $\mu \in \pi^{-1}(\rho(\lambda_0 + B_K^-))$ with $\nu = \lambda_0^{-1}\mu$. Then $\mu + B_K^- = \rho(\lambda_0 + B_K^-) = 1 + B_K^-$ and hence $|\mu - 1| < 1$ which implies that $|\mu| = 1$. Then $|\nu| = |\lambda_0|^{-1}|\mu| = |\lambda_0|^{-1} \leq 1$ and hence $\nu \in B_K$.
We prove that $\varphi_\nu = \rho$.

Let $\lambda \in K$. If $|\lambda| < |\lambda_0|$ then $\rho(\lambda + B_K^-) = 0 = \varphi_\nu(\lambda + B_K^-)$, since $|\nu| = |\lambda_0|^{-1}$. If $|\lambda| \geq |\lambda_0|$ then $\nu \in V_\lambda$, hence $\lambda\nu \in \pi^{-1}(\rho(\lambda + B_K^-))$ and thus

$$\varphi_\nu(\lambda + B_K^-) = \nu\lambda + B_K^- = \pi(\lambda\nu) = \rho(\lambda + B_K^-).$$

We see that $T : B_K \to K_1'$ is a bijective homomorphism and hence $K_1' \sim B_K$.
We now determine the dual of B_K.
We define $T : B_K' \to K_1$ by $T(f) = f(1) \quad (f \in B_K)$. It is not hard to verify that T is an injective homomorphism. Now T is also surjective since for every $y \in K_1$ the map $f_y : B_K \to K_1$ defined by $f_y(\lambda) = \lambda y \quad (\lambda \in B_K)$ is a continuous homomorphism and $T(f_y) = f_y(1) = y$.
We see that $B_K' \sim K_1$.

Chapter 5

Compact-like Properties for B_K-modules

5.1 B_K-modules of Finite Type and Locally Compactoids

B_K-modules of Finite Type

From locally convex K-vector space theory we know the following notion of a K-vector space of finite type, see [28], page 260.

5.1.1 Definition Let E be a K-vector space and let p be a vector space seminorm on E. Then p is called a *vector space seminorm of finite type* if $E/\mathrm{Ker}\, p$ is a finite-dimensional K-vector space.

5.1.2 Definition A locally convex K-vector space (E, τ) is called of *finite type* if every continuous vector space seminorm on E is of finite type.

If we extend this notion to locally convex B_K-modules we need just a slight modification.

5.1.3 Definition Let A be a B_K-module and let p be a seminorm on A. Then p is called (a *seminorm*) *of finite type* if $A/\mathrm{Ker}\, p \in \mathcal{F}_K$.

5.1.4 Definition A locally convex B_K-module (A, τ) is called (a *B_K-module*) *of finite type* if τ is generated by a collection of seminorms of finite type.

5.1.5 Remark Every member of $\mathcal{F}_K$ equipped with any locally convex topology is of finite type, since if $A \in \mathcal{F}_K$ then $A/\mathrm{Ker}\, p$ is a homomorphic image of A and hence $A/\mathrm{Ker}\, p \in \mathcal{F}_K$ for every seminorm p on A

To show why we have to modify the definition of K-vector space of finite type we prove the following.

5.1.6 Theorem *Let (A, τ) be a locally convex Hausdorff B_K-module such that every continuous seminorm on A is of finite type. Then $A \in \mathcal{F}_{\mathcal{K}}$.*

Proof: Suppose $A \notin \mathcal{F}_{\mathcal{K}}$. Then $A \neq \{0\}$. Let $x_1 \in A$, $x_1 \neq 0$.
The restriction of τ to $\mathrm{co}\{x_1\}$ is a Hausdorff locally convex topology. By using Corollary 4.1.6 and Corollary 3.3.16 we see that there exists a bounded norm $\| \ \|_1$ on $\mathrm{co}\{x_1\}$ such that $\tau|\mathrm{co}\{x_1\}$ equals the $\| \ \|_1$-topology.
By Theorem 3.4.20, $\| \ \|_1$ can be extended to a bounded τ-continuous seminorm p_1 on A. Then for every $\lambda \in B_K$: $p_1(\lambda x_1) = 0 \iff \lambda x_1 = 0$.
Furthermore, $\mathrm{Ker}\, p_1 \neq \{0\}$, for if $\mathrm{Ker}\, p_1 = \{0\}$, then $A = A/\mathrm{Ker}\, p_1 \in \mathcal{F}_{\mathcal{K}}$.
Let $x_2 \in \mathrm{Ker}\, p_1$, $x_2 \neq 0$.
Again by Corollary 4.1.6 there exists a bounded norm $\| \ \|_2$ on $\mathrm{co}\{x_2\}$ such that $\tau|\mathrm{co}\{x_2\}$ equals the $\| \ \|_2$-topology.
Therefore $\| \ \|_2$ can be extended to a bounded τ-continuous seminorm p_2 on A. Then for every $\lambda \in B_K$: $p_2(\lambda x_2) = 0 \iff \lambda x_2 = 0$.
Furthermore, $p_1 \vee p_2$ is a continuous seminorm and $\mathrm{Ker}\,(p_1 \vee p_2) \neq \{0\}$, for if $\mathrm{Ker}\,(p_1 \vee p_2) = \{0\}$, then $A = A/\mathrm{Ker}\,(p_1 \vee p_2) \in \mathcal{F}_{\mathcal{K}}$. Let $x_3 \in \mathrm{Ker}\,(p_1 \vee p_2)$, $x_3 \neq 0, \ldots$
Repeating this process we find bounded continuous seminorms $p_1, p_2, p_3, \ldots$ on A and $x_1, x_2, x_3, \ldots \in A$, with $x_n \neq 0$ for all $n \in \mathbb{N}$, such that for all $n \in \mathbb{N}$:

$$p_k(x_n) = 0 \text{ for all } k < n \text{ and for all } \lambda \in B_K\text{: } p_n(\lambda x_n) = 0 \iff \lambda x_n = 0.$$

We may assume $p_n \leq 1$, if necessary we take $p_n \wedge 1$ instead of p_n $(n \in \mathbb{N})$. Then for every $n \in \mathbb{N}$ the seminorm $q_n := \sup_{i \leq n} \frac{1}{i} p_i$ is continuous and $q_n \to \sup_{i \in \mathbb{N}} q_i$ uniformly on A, so $q := \sup_{i \in \mathbb{N}} q_i$ is a continuous seminorm on A. Hence, by assumption, $A/\mathrm{Ker}\, q \in \mathcal{F}_{\mathcal{K}}$. Thus there exist an $n \in \mathbb{N}$, an absolutely convex subset B of K^n and a surjective homomorphism φ from B to $A/\mathrm{Ker}\, q$. For every $i \in \mathbb{N}$ let $y_i \in B$ be such that $\varphi(y_i) = x_i + \mathrm{Ker}\, q$. Then $y_1, \ldots, y_{n+1}$ are linearly dependent. Hence, there exist $\lambda_1, \ldots, \lambda_{n+1} \in K$, with $\lambda_m \neq 0$ for some $m \in \{1, \ldots n+1\}$ such that $\lambda_1 y_1 + \cdots + \lambda_{n+1} y_{n+1} = 0$. Let $|\lambda_j| = \max\{|\lambda_i| : 1 \leq i \leq n+1\}$. Then $\frac{\lambda_i}{\lambda_j} \in B_K$ for every $i \in \{1, \ldots, n+1\}$ and $\frac{\lambda_1}{\lambda_j} y_1 + \cdots + \frac{\lambda_{n+1}}{\lambda_j} y_{n+1} = 0$. Then also

$$\frac{\lambda_1}{\lambda_j} x_1 + \cdots + \frac{\lambda_{n+1}}{\lambda_j} x_{n+1} + \mathrm{Ker}\, q = \varphi\Big(\frac{\lambda_1}{\lambda_j} y_1 + \cdots + \frac{\lambda_{n+1}}{\lambda_j} y_{n+1}\Big) = 0$$

and hence $\frac{\lambda_1}{\lambda_j} x_1 + \cdots + \frac{\lambda_{n+1}}{\lambda_j} x_{n+1} \in \mathrm{Ker}\, q$.
Let $k = \min\{i \mid 1 \leq i \leq n+1, \frac{\lambda_i}{\lambda_j} x_i \neq 0\}$. (Notice that the latter set is non-empty, because $\frac{\lambda_j}{\lambda_j} x_j = x_j \neq 0$.)
Then $\frac{\lambda_k}{\lambda_j} x_k + \cdots + \frac{\lambda_{n+1}}{\lambda_j} x_{n+1} \in \mathrm{Ker}\, q \subset \mathrm{Ker}\, p_k$.
Now also $\frac{\lambda_{k+1}}{\lambda_j} x_{k+1} + \cdots + \frac{\lambda_{n+1}}{\lambda_j} x_{n+1} \in \mathrm{Ker}\, p_k$, because $p_k(x_i) = 0$ for all $i > k$ and hence $\frac{\lambda_k}{\lambda_j} x_k \in \mathrm{Ker}\, p_k$. This is in contradiction with $\frac{\lambda_k}{\lambda_j} x_k \neq 0$.
Thus, $A \in \mathcal{F}_{\mathcal{K}}$. □

The following is an example of a B_K-module of finite type which is not a member of $\mathcal{F}_K$.

5.1.7 Example Let $A = B_K^{\mathbb{N}}$. It is clear that $A \notin \mathcal{F}_K$. Let $\| \ \|$ on A be defined by

$$\|(x_1, x_2, x_3, \dots)\| = \sup_{n \ge 1} \frac{1}{n} |x_n|.$$

It is easy to verify that $\| \ \|$ is a norm on A, that induces the product topology on A. Let p_n on A be defined by

$$p_n((x_1, x_2, x_3, \dots)) = |x_n| \qquad (n \ge 1).$$

Then, for every $n \ge 1$, p_n is a seminorm on A and $A/\operatorname{Ker} p_n \sim B_K \in \mathcal{F}_K$. Hence, p_n is a seminorm of finite type for every $n \ge 1$.
It is not hard to prove that the collection $\{p_n \mid n \ge 1\}$ generates the $\| \ \|$-topology. Hence, $(A, \| \ \|)$ is of finite type.

5.1.8 Proposition *A submodule of a B_K-module of finite type is of finite type.*

Proof: Let (A, τ) be a B_K-module of finite type and let B be a submodule of A. Let $\mathcal{P}$ be a collection seminorms of finite type on A generating τ. Let $\mathcal{P}' = \{p|B \mid p \in \mathcal{P}\}$. By using Proposition 3.4.11 we obtain that $\mathcal{P}'$ generates $\tau|B$. Let $q \in \mathcal{P}'$. Then there exists a $p \in \mathcal{P}$ such that $q = p|B$. Now $\varphi : B \to A/\operatorname{Ker} p$ defined by $\varphi(x) = x + \operatorname{Ker} p \quad (x \in B)$ is a homomorphism and $\operatorname{Ker} \varphi = \operatorname{Ker} p \cap B = \operatorname{Ker} q$. From Theorem 2.1.25 we obtain that there exists an injective homomorphism $B/\operatorname{Ker} q \to A/\operatorname{Ker} p$. Now $A/\operatorname{Ker} p \in \mathcal{F}_K$ and thus, by Proposition 2.2.36, also $B/\operatorname{Ker} q \in \mathcal{F}_K$. Hence, q is of finite type. We see $\mathcal{P}'$ is a collection seminorms of finite type on B generating $\tau|B$. Hence, $(B, \tau|B)$ is of finite type. □

5.1.9 Proposition *A product of B_K-modules of finite type is of finite type.*

Proof: Let I be an index set and for every $i \in I$ let (A_i, τ_i) be a B_K-module of finite type. Let $A = \Pi_{i \in I} A_i$ and let τ be the product topology on A.
Let, for every $i \in I$, $P_i : A \to A_i$ be the projection map on A_i and let $\mathcal{P}_i$ be a collection of seminorms of finite type on A_i generating τ_i. Let $\mathcal{P}$ be the collection $\{p \circ P_i \mid i \in I, \ p \in \mathcal{P}_i\}$. From Theorem 3.4.12 we obtain that $\mathcal{P}$ is a generating collection seminorms for τ. We prove that each $q \in \mathcal{P}$ is of finite type. To this end let $q \in \mathcal{P}$. Let $i \in I$ and $p \in \mathcal{P}_i$ be such that $q = p \circ P_i$. Now $\varphi : A \to A_i/\operatorname{Ker} p$ defined by $\varphi(x) = P_i(x) + \operatorname{Ker} p \quad (x \in A)$ is a surjective homomorphism and $\operatorname{Ker} \varphi = \operatorname{Ker} p \circ P_i = \operatorname{Ker} q$. From Theorem 2.1.25 we obtain that $A/\operatorname{Ker} q \sim A_i/\operatorname{Ker} p$. Now $A_i/\operatorname{Ker} p \in \mathcal{F}_K$ and hence also $A/\operatorname{Ker} q \in \mathcal{F}_K$. Thus q is of finite type.
We see that $\mathcal{P}$ is a collection seminorms of finite type on A that generates τ. Hence, (A, τ) is of finite type. □

5.1.10 Remark We can also prove here that a homomorphic image of a B_K-module of finite type is again a B_K-module of finite type, but this will follow from Theorem 5.1.16 and Proposition 5.1.15, see Corollary 5.1.17 (i).

5.1.11 Theorem (Representation theorem) *Let (A, τ) be a Hausdorff locally convex B_K-module. Then the following assertions are equivalent.*

(α) *(A, τ) is of finite type.*

(β) *(A, τ) is topologically embeddable in a product of normed B_K-modules of finite rank.*

(γ) *(A, τ) is topologically embeddable in a product of discrete torsion B_K-modules of finite rank.*

Proof: (α) $\Rightarrow$ (β): Let $\mathcal{P}$ be a collection seminorms of finite type that generates τ. In Proposition 3.4.16 we have seen that (A, τ) is topologically embeddable in $\prod_{p\in\mathcal{P}}(A/\mathrm{Ker}\,p, \overline{p})$, which is a product of normed modules in $\mathcal{F}_K$.
(β) $\Rightarrow$ (γ): Let $(A_\iota)_{\iota\in I}$ be a collection of normed modules in $\mathcal{F}_K$ such that (A, τ) is topologically embeddable in $\Pi_{\iota\in I}A_\iota$. For every $i \in I$ let $\|\ \|_\iota$ be a norm on A_ι that generates the topology on A_ι.
Let $A_\iota^n = A_\iota/\{x \in A_\iota \mid \|x\|_\iota \le \frac{1}{n}\}$ $(n \ge 1)$. In the same way as in the proof of Proposition 3.1.31 one can verify that A_ι is topologically embeddable in $\Pi_{n\ge1}A_\iota^n$ for each $i \in I$. Now A is topologically embeddable in $\Pi_{\iota\in I}A_\iota$ and $\Pi_{\iota\in I}A_\iota$ is topologically embeddable in $\Pi_{\iota\in I}\Pi_{n\ge1}A_\iota^n$. Hence, A is topologically embeddable in $\Pi_{\iota\in I}\Pi_{n\ge1}A_\iota^n$, which is a product of discrete torsion modules in $\mathcal{F}_K$.
(γ) $\Rightarrow$ (α): A discrete torsion module in $\mathcal{F}_K$ is of finite type. From Proposition 5.1.9 and Proposition 5.1.8 it follows that a product of B_K-modules of finite type is again of finite type and a submodule of a B_K-module of finite type is of finite type. Hence, (A, τ) is of finite type. □

5.1.12 Corollary *The completion of a Hausdorff B_K-module of finite type is again a B_K-module of finite type.*

Proof: Let (A, τ) be a Hausdorff B_K-module of finite type. By Theorem 5.1.11 there exist discrete torsion modules $(S_\iota)_{\iota\in I}$ in $\mathcal{F}_K$ and an injective, continuous and open homomorphism $i : A \to \prod_{\iota\in I} S_\iota$. Now $\prod_{\iota\in I} S_\iota$ is complete, as each S_ι is complete. The completion of A is topologically isomorphic to $\overline{i(A)} \subset \prod_{\iota\in I} S_\iota$. The product $\prod_{\iota\in I} S_\iota$ is of finite type since each S_ι is of finite type. Hence, $\overline{i(A)}$ is also of finite type, which implies that the completion of A is of finite type. □

5.1.13 Proposition *Let (A, τ) be a locally convex B_K-module of finite type. Then (A, τ) is of countable type.*

Proof: By the previous proposition we obtain that (A, τ) is topologically embeddable in a product of discrete torsion modules of finite rank, say $\prod_{\iota\in I} F_\iota$. By Proposition 2.2.44 we obtain that each F_ι is countably generated, and hence F_ι, provided with the discrete topology is of finite type (recall that the discrete topology is normable). From Proposition 4.2.16 we obtain that $\prod_{\iota\in I} F_\iota$ is of countable type and by Theorem 4.2.14 then also (A, τ) is of countable type. □

Locally Compactoids

5.1.14 Definition A locally convex B_K-module is called a *locally compactoid (B_K-module)* if $A/U \in \mathcal{F}_{\mathcal{K}}$ for every open submodule U of A.

5.1.15 Proposition *A continuous homomorphic image of a locally compactoid B_K-module is locally compactoid.*

Proof: Let (A, τ) be a locally compactoid B_K-module and let φ be a continuous surjective homomorphism from A to a locally convex B_K-module (B, σ). Let U be an open submodule of B. Then $\varphi^{-1}(U)$ is an open submodule of A. Thus $A/\varphi^{-1}(U) \in \mathcal{F}_{\mathcal{K}}$.
Now $\rho : A \to B/U$ defined by $\rho(x) = \varphi(x) + U \quad (x \in A)$ is a surjective homomorphism and $\operatorname{Ker} \rho = \varphi^{-1}(U)$. From Theorem 2.1.25 we obtain that $A/\varphi^{-1}(U) \sim B/U$ and hence also $B/U \in \mathcal{F}_K$. □

5.1.16 Theorem *Let (A, τ) be a Hausdorff locally convex B_K-module. Then: A is a B_K-module of finite type $\Longleftrightarrow$ A is a locally compactoid B_K-module.*

Proof: $\Rightarrow$) Let $\mathcal{P}$ be a collection seminorms of finite type generating τ. Let U be an open submodule of A. Then there exist $n \in \mathbb{N}$, $p_1, \dots, p_n \in \mathcal{P}$ and $\varepsilon_1, \dots, \varepsilon_n > 0$ such that $\{x \in A \mid p_i(x) < \varepsilon_i \quad (i = 1, \dots, n)\} \subset U$. The map $\varphi : A \to \prod_{i=1}^n A/\operatorname{Ker} p_i$ defined by

$$\varphi(x) = (x + \operatorname{Ker} p_1, \dots, x + \operatorname{Ker} p_n) \quad (x \in A)$$

is a homomorphism and $\operatorname{Ker} \varphi = \operatorname{Ker} p_1 \cap \dots \cap \operatorname{Ker} p_n$. By using Theorem 2.1.25 we obtain that there exists an injective homomorphism $A/(\operatorname{Ker} p_1 \cap \dots \cap \operatorname{Ker} p_n) \to \prod_{i=1}^n A/\operatorname{Ker} p_i$. Now $\prod_{i=1}^n A/\operatorname{Ker} p_i \in \mathcal{F}_K$ as $A/\operatorname{Ker} p_i \in \mathcal{F}_K$ for every $i \in \{1, \dots, n\}$. From Proposition 2.2.36 we obtain that $A/(\operatorname{Ker} p_1 \cap \dots \cap \operatorname{Ker} p_n) \in \mathcal{F}_K$. Now $\operatorname{Ker} p_1 \cap \dots \cap \operatorname{Ker} p_n \subset U$. Combining Proposition 2.1.27 and Proposition 2.2.26 we obtain that $A/U \in \mathcal{F}_K$. We see that (A, τ) is a locally compactoid B_K-module.
$\Leftarrow$) By Proposition 3.4.8, the collection $\{p_U\}_{U \text{ open submodule of } A}$ generates τ. Now $A/\operatorname{Ker} p_U = A/U \in \mathcal{F}_K$ for every open submodule U of A since (A, τ) is a locally compactoid B_K-module. Hence, (A, τ) is of finite type. □

5.1.17 Corollary

(i) *A continuous homomorphic image of a B_K-module of finite type is of finite type.*

(ii) *A submodule of a locally compactoid B_K-module is locally compactoid.*

(iii) *A product of locally compactoid B_K-modules is locally compactoid.*

In [26] and [21] the following definition of a locally compactoid absolutely convex subset of a K-vector space E is given.
An absolutely convex subset A of a K-vector space E is called a *locally compactoid in E* if for every zero neighbourhood U of E there exists a finite dimensional linear subspace D of E such that $A \subset U + D$.
The connection between the definition of a locally compactoid B_K-module and the above one is given in Theorem 5.1.18 and Theorem 5.1.19.

5.1.18 Theorem *Let (E, τ) be a locally convex K-vector space and let A be an absolutely convex subset of E, such that A is a locally compactoid in E. Then $(A, \tau|A)$ is a locally compactoid B_K-module.*

Proof: Let U be a $\tau|A$-open submodule of A. Then there exists a τ-open submodule V of E such that $U = V \cap A$. There exists a finite dimensional subspace D of E such that $A \subset V + D$. The inclusion map $i : A \to V + D$ induce a homomorphism $\varphi : A \to (V + D)/V$. Now $\operatorname{Ker}\varphi = A \cap V = U$ and hence, by applying Theorem 2.1.25, there exists an injective homomorphism $A/U \to (V + D)/V$. Hence, by Corollary 2.1.26, $(V + D)/V \sim D/(D \cap V)$. Now $D/(D \cap V)$ is a quotient of D and hence a member of $\mathcal{F}_K$. Then also $A/U \in \mathcal{F}_K$. We see that $(A, \tau|A)$ is locally compactoid. □

5.1.19 Theorem *Let (A, τ) be a Hausdorff locally convex B_K-module, that is topologically embeddable in a Hausdorff locally convex K-vector space E. If (A, τ) is a locally compactoid module, then there exists a locally convex space F with $E \subset F$ such that A is a locally compactoid in F.*

Proof: If the valuation on K is trivial then a locally compactoid is also compactoid. For that case we refer to Theorem 5.2.7. Now suppose K is non-trivial. Let $\mathcal{P}$ be the collection of all continuous (vector space) semi-norms on E. For every $p \in \mathcal{P}$ let $E_p = (E/\operatorname{Ker} p, \overline{p})$ and let $\hat{E_p}$ be the completion of E_p. Then $\hat{E_p}$ is a Banach space for every $p \in \mathcal{P}$. For $p \in \mathcal{P}$ let $\pi_p : E \to E_p$ be the quotient map and $i_p : E_p \to \hat{E_p}$ the inclusion map. Then for every $p \in \mathcal{P}$ the map $\varphi_p := i_p \circ \pi_p : E \to \hat{E_p}$ is a continuous homomorphism.

Let $p \in \mathcal{P}$ and let $t \in (0, 1]$. Let $e_0, e_1, e_2, \ldots$ be a t-orthogonal sequence in $\overline{\varphi_p(A)}$. We prove that $e_n \to 0$. To this end suppose that not $e_n \to 0$. That is to say that not $\overline{p}(e_n) \to 0$. We may assume that there exists an $\varepsilon > 0$ such that $\overline{p}(e_n) > \varepsilon$ for every $n \in \mathbb{N}$. Let $\mathrm{B}(0, t\varepsilon) = \{x \in \hat{E_p} \mid \overline{p}(x) < t\varepsilon\}$. Let $\pi_{t\varepsilon} : \hat{E_p} \to \hat{E_p}/\mathrm{B}(0, t\varepsilon)$ be the quotient map. Let $\rho : \overline{\varphi_p(A)} \to \hat{E_p}/\mathrm{B}(0, t\varepsilon)$ be defined as $\rho = \pi_{t\varepsilon} \circ \bar{\imath}$, where $\bar{\imath} : \overline{\varphi_p(A)} \to \hat{E_p}$ is the inclusion map. Let $V = \mathrm{B}(0, t\varepsilon) \cap \overline{\varphi_p(A)}$. Then $\operatorname{Im}\rho = \pi_{t\varepsilon}(\overline{\varphi_p(A)})$ and $\operatorname{Ker}\rho = V$. Hence, $\overline{\varphi_p(A)}/V \sim \pi_{t\varepsilon}(\overline{\varphi_p(A)})$. Now (A, τ) is a locally compactoid B_K-module and $\varphi_p : E \to \hat{E_p}$ is a continuous homomorphism, thus $\varphi_p(A)$ is a locally compactoid B_K-module and hence also $\overline{\varphi_p(A)}$ is locally compactoid, since $\overline{\varphi_p(A)}$ is topologically isomorphic to the completion of A, which is a locally compactoid B_K-module (see Theorem 5.1.16 and Corollary 5.1.12).

V is an open submodule of $\overline{\varphi_p(A)}$, so $\overline{\varphi_p(A)}/V \in \mathcal{F}_K$. Then also $\pi_{t\varepsilon}(\overline{\varphi_p(A)}) \in \mathcal{F}_K$.

There exist $n \in \mathbb{N}$, an absolutely convex subset B of K^n and a surjective homomorphism $\psi : B \to \pi_{t\varepsilon}(\overline{\varphi_p(A)})$.

Let $x_0, x_1, x_2, \ldots \in B$ such that $\psi(x_i) = \pi_{t\varepsilon}(e_i)$ for each $i \in \mathbb{N}$. There exist $\lambda_0, \ldots, \lambda_n \in K$ such that $\lambda_0 x_0 + \ldots + \lambda_n x_n = 0$ and $\lambda_i \neq 0$ for some $i \in \{0, \ldots n\}$. We may assume that $\lambda_0, \ldots, \lambda_n \in B_K$ and $\lambda_{i_0} = 1$ for some $i_0 \in \{0, \ldots, n\}$.

Then $\pi_{t\varepsilon}(\lambda_0 e_0 + \ldots + \lambda_n e_n) = \psi(\lambda_0 x_0 + \ldots + \lambda_n x_n) = \psi(0) = 0$. And hence $\overline{p}(e_{i_0}) \le \frac{1}{t}\overline{p}(\lambda_0 e_0 + \ldots + \lambda_n e_n) \le \frac{1}{t} t\varepsilon < \varepsilon$. This is in contradiction with $\overline{p}(e_m) > \varepsilon$ for every $m \in \mathbb{N}$. Hence, $e_n \to 0 \quad (n \to \infty)$.
By using Theorem 6p in [19] we obtain that $\overline{\varphi_p(A)}$ is a locally compactoid subset of $\hat{E_p}$ and hence so is $\varphi_p(A)$.
The map $\Phi : E \to \prod_{p\in\mathcal{P}} \hat{E_p}$ defined by $(\Phi(x))(p) = \varphi_p(x) \quad (p \in \mathcal{P}, x \in E)$ is a linear homeomorphism from E in $\prod_{p\in\mathcal{P}} \hat{E_p}$.
Now $\varphi_p(A)$ is a locally compactoid subset of $\hat{E_p}$ for every $p \in \mathcal{P}$, thus $\prod_{p\in\mathcal{P}} \varphi_p(A)$ is a locally compactoid subset of $\prod_{p\in\mathcal{P}} \hat{E_p}$ (see Proposition 1.2 in [21]). It is not hard to see that A is topologically embeddable in $\prod_{p\in\mathcal{P}} \varphi_p(A)$. Hence, A is also a locally compactoid in $\prod_{p\in\mathcal{P}} \hat{E_p}$. □

5.1.20 Remark An absolutely convex subset of a locally convex K-vector space E that is a locally compactoid B_K-module need not be locally compactoid in E. In [21] Schikhof gives a counterexample.

Now we will mention some results on metrizable locally compactoids. The following theorem is a generalization of Lemma 1.3 in [26]

5.1.21 Theorem *Let (A, τ) be a metrizable, complete locally compactoid B_K-module. Let $\lambda \in B_K^-$. Then there exist submodules $F_1, F_2, F_3, \ldots$ of rank ≤ 1 of A with $\lim_{n\to\infty} F_n = \{0\}$ such that $\lambda A \subset \overline{\mathrm{co}}\{F_1, F_2, F_3, \ldots\} \subset A$.*

Proof: If the valuation on K is trivial then a locally compactoid is also compactoid. For that case we refer to Proposition 5.2.10. Now assume that $|K|$ is non-trivial. A is metrizable thus A has a countable base of zero neighbourhoods $U_1, U_2, U_3, \ldots$, consisting of open submodules of A. Then for every $n \in \mathbb{N}$ the quotient A/U_n is a discrete torsion module in $\mathcal{F}_K$. By Proposition 3.1.31 there exists an injective continuous and open homomorphism $i : A \to \prod_{n\in\mathbb{N}} A/U_n$.
For each $n \in \mathbb{N}$ let $k_n = \operatorname{rank} A/U_n$, let C_n be an absolutely convex subset of K^{k_n} and let $\varphi_n : C_n \to A/U_n$ be a surjective homomorphism. Let $C = \prod_{n\in\mathbb{N}} C_n$, provided with the product topology. From Proposition 4.1.7 we obtain that φ_n is a quotient map for every $n \in \mathbb{N}$. By Proposition 3.1.18 there exists a quotient map $\varphi : C \to \prod_{n\in\mathbb{N}} A/U_n$. By Proposition 3.1.19 we may consider C as an absolutely convex subset of $K^{\mathbb{N}}$ provided with the product topology. Now C is closed for C_n is closed for every $n \in \mathbb{N}$ and hence C is a complete locally compactoid in $K^{\mathbb{N}}$.
Now $\varphi^{-1}(i(A))$ is closed in C and is therefore a complete locally compactoid in $K^{\mathbb{N}}$. From Lemma 1.3 of [26] we obtain that there exist absolutely convex subsets $X_1, X_2, X_3, \ldots$ of rank ≤ 1 of $K^{\mathbb{N}}$ with $\lim_{n\to\infty} X_n = \{0\}$ such that $\varphi^{-1}(i(A)) \subset \overline{\sum X_n} \subset \lambda^{-1}\varphi^{-1}(i(A))$. Then

$$\lambda\varphi^{-1}(i(A)) \subset \overline{\sum \lambda X_n} \subset \varphi^{-1}(i(A)).$$

Let $G_n = \varphi(\lambda X_n) \quad (n \in \mathbb{N})$. Then $\operatorname{rank} G_n = 1$ for every n, $\lim_{n\to\infty} G_n = 0$ and $i(\lambda A) \subset \overline{\mathrm{co}}\{G_1, G_2, G_3, \ldots\} \subset i(A)$. We prove the last assertion. To

this end let $z \in i(\lambda A)$. Then there exists a $y \in A$ such that $i(\lambda y) = z$. Let $w \in \overline{\varphi^{-1}(i(A))}$ be such that $\varphi(w) = i(y)$. Then $\varphi(\lambda w) = z$. Now $\lambda w \in \overline{\sum \lambda X_n}$ and hence there exist $w_1, w_2, w_3, \ldots \in \sum \lambda X_n$ such that $\lim_{n\to\infty} w_n = \lambda w$. Then $\varphi(w_1), \varphi(w_2), \varphi(w_3), \ldots \in \mathrm{co}\{G_1, G_2, G_3, \ldots\}$ and $z = \varphi(\lambda w) = \lim_{n\to\infty} \varphi(w_n) \in \overline{\mathrm{co}}\{G_1, G_2, G_3, \ldots\}$.
We see that $i(\lambda A) \subset \overline{\mathrm{co}}\{G_1, G_2, G_3, \ldots\}$. As $i(A)$ is closed we obtain that $\overline{\mathrm{co}}\{G_1, G_2, G_3, \ldots\} \subset i(A)$.
Let $F_n = i^{-1}(G_n)$ $(n \in \mathbb{N})$. Then $\operatorname{rank} F_n = 1$ for every $n \in \mathbb{N}$, $\lim_{n\to\infty} F_n = 0$ and $\lambda A \subset \overline{\mathrm{co}}\{F_1, F_2, F_3, \ldots\} \subset A$. □

5.1.22 Corollary *Let A be a complete metrizable locally compactoid B_K-module and let U be an open submodule of A. Let $\lambda \in B_{\bar{K}}$. Then there exist submodules $F_1, \ldots, F_n$ of rank ≤ 1 of A such that $\lambda A \subset (F_1 + \cdots + F_n) + U \subset A$.*

Proof: There exist rank 1 submodules $F_1, F_2, F_3, \ldots$ of A with $\lim_{n\to\infty} F_n = 0$ such that $\lambda A \subset \overline{\mathrm{co}}\{F_1, F_2, F_3, \ldots\} \subset A$.
As $F_k \to 0$ there exists an $n \in \mathbb{N}$ such that $F_k \subset U$ for every $k > n$.
Let $x \in \overline{\mathrm{co}}\{F_1, F_2, F_3, \ldots\}$. Then there exists a $y \in \mathrm{co}\{F_1, F_2, F_3, \ldots\}$ such that $x - y \in U$. There exist $m \in \mathbb{N}$ and $z_1 \in F_1, \ldots, z_m \in F_m$ such that $y = z_1 + \cdots + z_m$. If $m \leq n$ then $y \in F_1 + \cdots + F_n$ and hence

$$x = y + (x - y) \in (F_1 + \cdots + F_n) + U.$$

If $m > n$ then $z_{n+1}, \ldots, z_m \in U$ and

$$\begin{aligned} x = (z_1 + \cdots + z_n) + (z_{n+1} + \cdots + z_m) + (x - y) \in \\ (F_1 + \cdots + F_n) + U + U = (F_1 + \cdots + F_n) + U. \end{aligned}$$

We see that $\overline{\mathrm{co}}\{F_1, F_2, F_3, \ldots\} \subset (F_1 + \cdots + F_n) + U$.
Then $\lambda A \subset (F_1 + \cdots + F_n) + U \subset A$. □

5.1.23 Remark In Theorem 5.1.21 and Corollary 5.1.22 we can not omit the completeness of A. In Example 3.6 of [21] a metrizable locally compactoid B_K-module A is presented for which the conclusions of Theorem 5.1.21 and Corollary 5.1.22 are not true.

5.2 Compactoids

5.2.1 Definition A locally convex B_K-module (A, τ) is called a *compactoid* (B_K*-module*) if $A/U \in \mathcal{B}_K$ for every open submodule U of A.

5.2.2 Remark If the valuation on K is trivial then $\mathcal{F}_K = \mathcal{B}_K$ and hence every locally compactoid is also a compactoid.

One might think that, for trivially valued K, the notion of compactoid is trivial (that is to say that every compactoid is a member of $\mathcal{B}_K$) but that is not true. In fact, in Example 5.1.7 we have seen a a B_K-module of finite type that is not a member of $\mathcal{F}_K$, even if K is trivial. If K is trivial this module is also a compactoid.

5.2.3 Proposition *Let (A, τ) be a locally convex B_K-module. Then: (A, τ) is a bounded locally compactoid $\iff$ (A, τ) is a compactoid.*

Proof: $\Rightarrow$) Let U be an open submodule of A. We prove that $A/U \in \mathcal{B}_K$. As (A, τ) is a locally compactoid we obtain that $A/U \in \mathcal{F}_K$. Since (A, τ) is bounded we obtain, by using Proposition 4.3.9, that A/U is bounded with respect to the quotient topology on A/U (which is the discrete topology). Proposition 4.3.12 then implies that $A/U \in \mathcal{B}_K$.
$\Leftarrow$) By definition (A, τ) is a locally compactoid B_K-module. We now prove that A is bounded. Let C be a base of zero neighbourhoods of τ consisting of submodules of A. Then $A/U \in \mathcal{B}_K$ for every $U \in C$, since (A, τ) is a compactoid. By Proposition 3.1.31, (A, τ) is topologically embeddable in $\prod_{U \in C} A/U$, and the latter set is bounded since each A/U is bounded (see Proposition 4.3.10). Hence, by Proposition 4.3.6, (A, τ) is bounded. □

The following proposition is a consequence of Proposition 5.2.3.

5.2.4 Proposition

1. *A submodule of a compactoid B_K-module is compactoid.*

2. *A continuous homomorphic image of a compactoid B_K-module is compactoid.*

3. *A product of compactoid B_K-modules is compactoid.*

4. *The completion of a compactoid B_K-module is compactoid.*

Proof: 1. Combine Proposition 5.2.3, Corollary 5.1.17 and Proposition 4.3.6.
2. Combine Proposition 5.2.3, Proposition 5.1.15 and proposition 4.3.9.
3. Combine Proposition 5.2.3, Corollary 5.1.17 and Proposition 4.3.10.
4. Combine Proposition 5.2.3, Theorem 5.1.16, Corollary 5.1.12 and Proposition 4.3.8. □

5.2.5 Theorem *Let (A, τ) be a Hausdorff locally convex B_K-module. Then: (A, τ) is a compactoid $\iff$ (A, τ) is topologically embeddable in a product of finitely generated discrete torsion modules.*

Proof: $\Rightarrow$) Let C be the collection of all open submodules of A.
For each $U \in C$ the quotient A/U is a discrete torsion module in $\mathcal{B}_K$ and hence, by Proposition 2.2.40, there exists a finitely generated discrete torsion B_K-module A_U such that $A/U \subset A_U$.
From Proposition 3.1.17 we obtain that $\prod_{U \in C} A/U$ is topologically embeddable in $\prod_{U \in C} A_U$. As (A, τ) is topologically embeddable in $\prod_{U \in C} A/U$, we obtain that (A, τ) is also topologically embeddable in $\prod_{U \in C} A_U$.
$\Leftarrow$) Let $(B_\iota)_{\iota \in I}$ be a collection of finitely generated discrete torsion modules in $\mathcal{B}_K$ such that (A, τ) is topologically embeddable in $B := \prod_{\iota \in I} B_\iota$. Then each B_ι is a compactoid and hence, by 3. of the previous proposition, also B is a compactoid. As (A, τ) is topologically embeddable in B it follows from 1. of the same proposition that also (A, τ) is a compactoid B_K-module. □

5.2.6 Remark If K is spherically complete then, by Theorem 2.2.21, every finitely generated torsion B_K-module is a product of 1-generated torsion B_K-modules. Thus, for a Hausdorff locally convex B_K-module over a spherically complete K we have even the following.
(A,τ) is compactoid $\Longleftrightarrow$ (A,τ) is topologically embeddable in a product of discrete 1-generated torsion B_K-modules.

In the following theorem the relation between the notion of an absolutely convex compactoid subset of a K-vector space E and that of a compactoid B_K-module is given.

5.2.7 Theorem *Let A be an absolutely convex subset of a Hausdorff locally convex K-vector space (E,τ). Then:*
$(A,\tau|A)$ is a compactoid B_K-module $\Longleftrightarrow$ for every zero neighbourhood U of E there exist $x_1,\dots,x_n \in E$ such that $A \subset U + \operatorname{co}\{x_1,\dots,x_n\}$ (i.e. A is a compactoid in E).

Proof: $\Rightarrow$) Let U be a zero neighbourhood in E. There exists an open sumodule V of E such that $V \subset U$. If $|K|$ is discrete let $\lambda = 1$, if $|K|$ is dense let $\lambda \in B_K^- \setminus \{0\}$. As $x \mapsto \lambda x$ is a homeomorphism in (E,τ) we obtain that also λV is an open submodule of E. Let $W = \lambda V \cap A$. Then W is an open submodule of A and hence $A/W \in \mathcal{B}_K$. By Proposition 2.2.41 there exists a finitely generated module X such that $\lambda(A/W) \subset X \subset (A/W)$. Let $e_1,\dots,e_n \in A/W$ such that $X = \operatorname{co}\{e_1,\dots,e_n\}$. Let $x_1,\dots,x_n \in A$ such that $e_i = x_i + W$ for $i \in \{1,\dots,n\}$. Then $\lambda A \subset W + \operatorname{co}\{x_1,\dots,x_n\}$. Hence,

$$\begin{aligned} A &\subset \lambda^{-1}W + \operatorname{co}\{\lambda^{-1}x_1,\dots,\lambda^{-1}x_n\} \subset V \cap \lambda^{-1}A + \operatorname{co}\{\lambda^{-1}x_1,\dots,\lambda^{-1}x_n\} \\ &\subset U + \operatorname{co}\{\lambda^{-1}x_1,\dots,\lambda^{-1}x_n\}. \end{aligned}$$

$\Leftarrow$) Let U be an open submodule of A. Then there exists an open submodule V of E such that $U = V \cap A$. There exist $x_1,\dots,x_n \in E$ such that $A \subset V + \operatorname{co}\{x_1,\dots,x_n\}$. The inclusion map $A \to V + \operatorname{co}\{x_1,\dots,x_n\}$ induces a homomorphism $\varphi : A \to (V + \operatorname{co}\{x_1,\dots,x_n\})/V$. Now $\operatorname{Ker}\varphi = U$, thus there exists an injective homomorphism $\tilde{\varphi} : A/U \to (V + \operatorname{co}\{x_1,\dots,x_n\})/V$. Now

$$(V + \operatorname{co}\{x_1,\dots,x_n\})/V \sim (\operatorname{co}\{x_1,\dots,x_n\}/(\operatorname{co}\{x_1,\dots,x_n\} \cap V),$$

thus $(V + \operatorname{co}\{x_1,\dots,x_n\})/V$ is a quotient of $\operatorname{co}\{x_1,\dots,x_n\}$ and hence a member of $\mathcal{B}_K$. As A/U is embeddable in $(V + \operatorname{co}\{x_1,\dots,x_n\})/V$ we obtain that also $A/U \in \mathcal{B}_K$.
We see $(A,\tau|A)$ is a compactoid B_K-module. □

In [14] Katsaras proved the following theorem.
Let A be an absolutely convex compactoid subset of a locally convex Hausdorff K-vector space (E,τ). If $|K|$ is discrete let $\lambda = 1$, if $|K|$ is dense let $\lambda \in K$, $|\lambda| > 1$. Then for every zero neighbourhood U of E there exist $x_1,\dots,x_n \in \lambda A$ such that $A \subset U + \operatorname{co}\{x_1,\dots,x_n\}$.
A proof of this theorem can also be obtained from the proof of Theorem 5.2.7. For compactoid B_K-modules we modify Katsaras' Theorem as follows.

5.2.8 Proposition *Let (A, τ) be a compactoid B_K-module. If $|K|$ is discrete let $\lambda = 1$, if $|K|$ is dense let $\lambda \in B_K^-$. Let U be an open submodule of A. Then there exist $x_1, \dots, x_n \in A$ such that $\lambda A \subset U + \mathrm{co}\{x_1, \dots, x_n\} \subset A$.*

Proof: By Proposition 2.2.41 there exists a finitely generated submodule Y of A/U such that $\lambda(A/U) \subset Y \subset A/U$. Let $y_1, \dots, y_n \in A/U$ be such that $Y = \mathrm{co}\{y_1, \dots, y_n\}$. Let $x_1, \dots, x_n \in A$ be such that $y_i = x_i + U$ for all $i \in \{1, \dots, n\}$. Then $\lambda A \subset U + \mathrm{co}\{x_1, \dots, x_n\} \subset A$. □

5.2.9 Remark The converse of Proposition 5.2.8 is not true: let the valuation on K be dense. If (A, τ) is a locally convex B_K-module such that for open submodule U and every $\lambda \in B_K^-$ there exist $x_1, \dots, x_n \in A$ such that $\lambda A \subset U + \mathrm{co}\{x_1, \dots, x_n\}$, then A need not be compactoid.
For example, let $A = (B_K/B_K^-)^{\mathbb{N}}$, provided with the discrete topology, observe that the discrete topology is locally convex since $\{0\}$ is absorbing. For every open submodule U of A and every $\lambda \in B_K^-$ we have that $\lambda A = \{0\} \subset U$. But, A is not a compactoid, since $\{0\}$ is open and $A/\{0\} = A$, and $A \notin \mathcal{B}_K$, which is easy to verify.

The counterpart of Theorem 5.1.21 for compactoid B_K-modules is also true. We even do not need the completeness of the compactoid as we see in the following proposition.

5.2.10 Proposition *Let (A, τ) be a metrizable compactoid B_K-module. If $|K|$ is discrete let $\lambda = 1$, if $|K|$ is dense let $\lambda \in B_K$, $|\lambda| < 1$. Then there exist $x_1, x_2, x_3, \dots \in A$ with $\lim_{n\to\infty} x_n = 0$ such that $\lambda A \subset \overline{\mathrm{co}}\{x_1, x_2, x_3, \dots\} \subset A$.*

Proof: Suppose $|K|$ is dense. Let $A = U_0 \supset U_1 \supset U_2 \supset \dots$ be a base of zero neighbourhoods consisting of submodules of A. Let $\lambda_1, \lambda_2, \lambda_3, \dots \in B_K^-$ such that $|\lambda_1||\lambda_2||\lambda_3| \cdots > |\lambda|$. There exist $x_1, \dots, x_{n_1} \in A$ such that

$$\lambda_1 A \subset U_1 + \mathrm{co}\,\{x_1, \dots, x_{n_1}\} \subset A.$$

Now U_1 is a compactoid B_K-module and $U_2 = U_2 \cap U_1$ is an open submodule of U_1, hence there exists $x_{n_1+1}, \dots, x_{n_2} \in U_1$ such that

$$\lambda_2 U_1 \subset U_2 + \mathrm{co}\,\{x_{n_1+1}, \dots, x_{n_2}\} \subset U_1.$$

Then

$$\lambda_1\lambda_2 A \subset \lambda_2 U_1 + \mathrm{co}\,\{x_1, \dots, x_{n_1}\} \subset U_2 + \mathrm{co}\,\{x_1, \dots, x_{n_1}, x_{n_1+1}, \dots, x_{n_2}\} \subset A.$$

Continuing this process we find $0 = n_0 \le n_1 \le n_2 \le n_3, \dots \in \mathbb{N}$ and $x_1, x_2, x_3, \dots \in A$ such that $x_{n_k+1}, \dots, x_{n_{k+1}} \in U_k$ for all $k \in \mathbb{N}$ and

$$\lambda_1\lambda_2 \cdots \lambda_k A \subset U_k + \mathrm{co}\,\{x_1, \dots, x_{n_k}\} \subset A \quad (k \ge 1).$$

Then $\lim_{n\to\infty} x_n = 0$ and

$$\lambda A \subset U_k + \mathrm{co}\,\{x_1, x_2, x_3, \dots\} \subset A \quad (k \ge 1).$$

Hence,

$$\lambda A \subset \overline{\mathrm{co}}\{x_1, x_2, x_3, \dots\} \subset A.$$

If the valuation on K is discrete then (A, τ) is a pure compactoid module (see Proposition 5.2.15). For that case we refer to Theorem 5.2.20. □

5.2.11 Theorem *Let (A, τ) be a Hausdorff compactoid B_K-module. Then there exists a locally convex K-vector space of finite type E, an absolutely convex compactoid subset S of E, with $[S] = E$ and a quotient map $\varphi : S \to A$.*

Proof: 1. From the proof of Theorem 5.2.5 we know that there exist a collection $(C_\iota)_{\iota\in I}$ of discrete torsion modules in $\mathcal{B}_K$ and a homeomorphism j from (A, τ) in $\prod_{\iota\in I} C_\iota$, where $\prod_{\iota\in I} C_\iota$ is provided with the product topology.
For every $i \in I$ let $n_\iota \in \mathbb{N}$, S_ι a bounded absolutely convex subset of K^{n_ι} and $\varphi_\iota : S_\iota \to C_\iota$ a surjective homomorphism. Each S_ι is provided with the unique locally convex Hausdorff topology (see Proposition 4.1.4). From Proposition 4.1.7 it follows that every φ_ι is continuous. As the topology on each C_ι is discrete we obtain that each φ_ι is a quotient map. By using Proposition 3.1.18 we obtain that the map $\Phi : \prod_{\iota\in I} S_\iota \to \prod_{\iota\in I} C_\iota$ defined by

$$(\Phi(x))(i) = \varphi_\iota(x(i)) \quad (i \in I) \quad (x = (x(i))_{\iota\in I} \in \prod_{\iota\in I} S_\iota)$$

is a quotient map.
2. We consider j as a homeomorphism $A \to j(A)$. Then $j^{-1} \circ (\Phi|\Phi^{-1}(j(A)))$ is a quotient map $\Phi^{-1}(j(A)) \to A$.
For every $i \in I$ the set S_ι is a compactoid in K^{n_ι} and hence, $\prod_{\iota\in I} S_\iota$ is a compactoid subset of $\prod_{\iota\in I} K^{n_\iota}$. Now $\Phi^{-1}(j(A)) \subset \prod_{\iota\in I} S_\iota$ and hence also $\Phi^{-1}(j(A))$ is a compactoid subset of $\prod_{\iota\in I} K^{n_\iota}$.
3. Let $E = [\Phi^{-1}(j(A))] \subset \prod_{\iota\in I} K^{n_\iota}$. Then E is the linear span of a compactoid subset and hence, by Corollary 1.5 in [21], E is of finite type. Moreover, $\Phi^{-1}(j(A))$ is a compactoid subset of E and the homomorphism $j^{-1} \circ (\Phi|\Phi^{-1}(j(A))) : \Phi^{-1}(j(A)) \to A$ is a quotient map. □

5.2.12 Theorem *Every Hausdorff compactoid B_K-module is topologically embeddable in a divisible locally compactoid B_K-module.*

Proof: Let (A, τ) be a compactoid B_K-module. By Theorem 5.2.11 there exists a locally convex K-vector space (E, τ) of finite type, an absolutely convex subset S of E such that $[S] = E$ and a quotient map $\varphi : S \to A$.
There exists an injective homomorphism $j : S/\mathrm{Ker}\,\varphi \to E/\mathrm{Ker}\,\varphi$ such that the following diagram is commutative.

$$\begin{tikzcd}
S \arrow[r, "i"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\
S/\mathrm{Ker}\,\varphi \arrow[r, "j"'] & E/\mathrm{Ker}\,\varphi
\end{tikzcd}$$

(Here i, π are the natural maps.)
Then j is of course a homeomorphism in $E/\mathrm{Ker}\,\varphi$. Hence $S/\mathrm{Ker}\,\varphi$ is topologically isomorphic to $j(S/\mathrm{Ker}\,\varphi)$. As (A, τ) is homeomorphic to $S/\mathrm{Ker}\,\varphi$, provided with the quotient topology, we obtain that (A, τ) is embeddable in $E/\mathrm{Ker}\,\varphi$.
It is obvious that (E, τ) is also a B_K-module of finite type and hence a locally compactoid. Then, by Proposition 5.1.15, also $E/\mathrm{Ker}\,\varphi$ is a locally compactoid B_K-module. That $E/\mathrm{Ker}\,\varphi$ is divisible is easy to check. □

5.2.13 Remark Let A, E and φ be as in the above proof. If we view A as a subset of $E/\mathrm{Ker}\,\varphi$ we can prove the following Katsaras-like theorem.
If $|K|$ is discrete let $\lambda = 1$ and if $|K|$ is dense let $\lambda \in K$ with $|\lambda| > 1$. Then for every open submodule U of $E/\mathrm{Ker}\,\varphi$ there exist $x_1, \dots, x_n \in \lambda A$ such that $A \subset U + \mathrm{co}\{x_1, \dots, x_n\}$.

Pure Compactoids

5.2.14 Definition Let (A, τ) be a locally convex B_K-module. A is called a *pure compactoid* (B_K-module) if for every open submodule U of A the quotient A/U is a finitely generated B_K-module. Clearly a pure compactoid is a compactoid.

5.2.15 Proposition *Let the valuation on K be discrete. Then every compactoid B_K-module is a pure compactoid.*

Proof: Let (A, τ) be a compactoid B_K-module and let U be an open submodule of A. Then $A/U \in \mathcal{B}_K$. From Proposition 2.2.38 we obtain that A/U is finitely generated. □

5.2.16 Proposition

1. *A product of pure compactoid B_K-modules is a pure compactoid B_K-module.*

2. *A continuous homomorphic image of a pure compactoid B_K-module is a pure compactoid B_K-module.*

Proof: 1. Let I be an index set and for each $i \in I$ let (A_i, τ_i) be a pure compactoid B_K-module. Let $A = \prod_{i \in I} A_i$ and let τ be the product topology on A.
Let U be an open submodule of A. Then there exist $n \in \mathbb{N}$, $i_1, \dots, i_n \in I$ and open submodules $U_1 \subset A_{i_1}, \dots, U_n \subset A_{i_n}$ such that $\bigcap_{j=1}^n P_{i_j}^{-1}(U_j) \subset U$. (Here $P_{i_j} : A \to A_{i_j}$ is the projection on A_{i_j} for each $j \in \{1, \dots, n\}$.) Now A_{i_j}/U_j is finitely generated for every $j \in \{1, \dots, n\}$ and hence so is $\prod_{j=1}^n A_{i_j}/U_j$. The map $\rho : A \to \prod_{j=1}^n A_{i_j}/U_j$, defined by

$$\rho(x) = (P_{i_1}(x) + U_1, \dots, P_{i_n}(x) + U_n) \quad (x \in A),$$

is a surjective homomorphism and $\mathrm{Ker}\,\rho \subset U$. Combining Theorem 2.1.25 and Proposition 2.1.27 we obtain that there exists a surjective homomorphism $\prod_{j=1}^n A_{i_j}/U_j \to A/U$. Hence, also A/U is finitely generated. We see that (A, τ) is a pure compactoid B_K-module.
2. Let (A, τ) be a pure compactoid B_K-module. Let (B, σ) be a locally convex B_K-module and let $\varphi : A \to B$ be a continuous surjective homomorphism. Let U be an open submodule of B. Then φ induces a surjective homomorphism $A/\varphi^{-1}(U) \to B/U$. Since $\varphi^{-1}(U)$ is an open submodule of A we obtain that $A/\varphi^{-1}(U)$ is finitely generated. Then B/U, as a homomorphic image of $A/\varphi^{-1}(U)$, is also finitely generated.
We see that (B, σ) is a pure compactoid B_K-module. □

5.2.17 Proposition *Let (A,τ) be a locally convex B_K-module. Then: (A,τ) is a pure compactoid $\Longleftrightarrow$ For every open submodule U of A there exist $x_1,\dots,x_n \in A$ such that $A = U + \mathrm{co}\{x_1,\dots,x_n\}$.*

Proof: $\Rightarrow$) Let U be an open submodule of A. Then A/U is finitely generated. Let $e_1,\dots e_n \in A/U$ such that $A/U = \mathrm{co}\{e_1,\dots,e_n\}$. Let $x_1,\dots,x_n \in A$ such that $e_i = x_i + U \quad (i = 1,\dots,n)$. Then $A = U + \mathrm{co}\{x_1,\dots,x_n\}$.
$\Leftarrow$) Let U be an open submodule of A. Then there exist $x_1,\dots,x_n \in A$ such that $A = U + \mathrm{co}\{x_1,\dots,x_n\}$. Then $A/U = \mathrm{co}\{x_1 + U,\dots,x_n + U\}$. Hence, A/U is finitely generated. □

5.2.18 Remark Let (A,τ) be a locally convex K-vector space. Let A be an absolutely convex subset of E. From the previous proposition it follows that *$(A,\tau|A)$ is a pure compactoid B_K-module $\Longleftrightarrow$ A is a pure compactoid subset of E.*
Recall that A is called a pure compactoid subset of (E,τ) if for every zero neighbourhood U of E there exist an $n \in \mathbb{N}$ and $x_1,\dots,x_n \in A$ such that $A \subset U + \mathrm{co}\{x_1,\dots,x_n\}$.

5.2.19 Proposition *Let (A,τ) be a locally convex B_K-module. Then: A is a pure compactoid B_K-module $\Longleftrightarrow$ The completion $\widehat{A}$ of A is a pure compactoid B_K-module.*

Proof: $\Rightarrow$) Let U be an open submodule of $\widehat{A}$. Then $V := U \cap A$ is an open submodule in A. According to Theorem 5.2.17 there exist $x_1,\dots,x_n \in A$ such that $A = V + \mathrm{co}\{x_1,\dots,x_n\}$. Then $\widehat{A} \subset U + \mathrm{co}\{x_1,\dots,x_n\}$. To show this, let $y \in \widehat{A}$. Then there exists an $x \in A$ such that $y - x \in U$. There exist $v \in V$ and $\lambda_1,\dots,\lambda_n \in B_K$ such that $x = v + (\lambda_1 x_1 + \cdots + \lambda_n x_n)$. Then $y - x = y - (\lambda_1 x_1 + \cdots + \lambda_n x_n) - v \in U$ and hence

$$y - (\lambda_1 x_1 + \cdots + \lambda_n x_n) \in U + V = U.$$

Thus $y \in U + (\lambda_1 x_1 + \cdots + \lambda_n x_n) \subset U + \mathrm{co}\{x_1,\dots,x_n\}$.
$\Leftarrow$) Let U be an open submodule of A. Then there exists an open submodule V of $\widehat{A}$ such that $U = V \cap A$. There exist $x_1,\dots,x_n \in \widehat{A}$ such that $\widehat{A} \subset V + \mathrm{co}\{x_1,\dots,x_n\}$. Let $y_1,\dots,y_n \in A$ such that $x_i - y_i \in V$ for $i = 1,\dots,n$. Then also $\widehat{A} \subset V + \mathrm{co}\{y_1,\dots,y_n\}$. To show this, let $x \in \widehat{A}$. Then there exist $v \in V$ and $\lambda_1,\dots\lambda_n \in B_K$ such that $x = v + \sum_{i=1}^n \lambda_i x_i$. Then

$$\sum_{i=1}^n \lambda_i x_i - \sum_{i=1}^n \lambda_i y_i = \sum_{i=1}^n \lambda_i (x_i - y_i) \in V$$

and

$$x = v + (\sum_{i=1}^n \lambda_i x_i - \sum_{i=1}^n \lambda_i y_i) + \sum_{i=1}^n \lambda_i y_i \in V + \mathrm{co}\{y_1,\dots,y_n\}$$

Then $A \subset V \cap A + \mathrm{co}\{y_1,\dots,y_n\} = U + \mathrm{co}\{y_1,\dots,y_n\}$. □

We now prove a proposition on metrizable pure compactoid modules.

5.2.20 Proposition *Let (A,τ) be a metrizable pure compactoid B_K-module. Then there exist $x_1,x_2,x_3,\ldots \in A$ with $\lim_{n\to\infty} x_n = 0$ such that $A = \overline{\mathrm{co}}\{x_1,x_2,x_3,\ldots\}$.*

Proof: Let $A = U_0 \supset U_1 \supset U_2 \supset \ldots$ be a base of zero neighbourhoods consisting of submodules of A. There exist $n_1 \in \mathbb{N}$ and $x_1,\ldots,x_{n_1} \in A$ such that

$$A = U_1 + \mathrm{co}\,\{x_1,\ldots,x_{n_1}\}.$$

Let $n_2 \geq n_1$ and $y_{n_1+1},\ldots,y_{n_2} \in A$ such that $A = U_2 + \mathrm{co}\,\{y_{n_1+1},\ldots,y_{n_2}\}$. There exist $x_{n_1+1},\ldots,x_{n_2} \in U_1$ and $z_{n_1+1},\ldots,z_{n_2} \in \mathrm{co}\,\{x_1,\ldots,x_{n_1}\}$ such that $y_\iota = x_\iota + z_\iota$ for $i = n_1+1,\ldots,n_2$. Then

$$A = U_2 + \mathrm{co}\,\{x_1,\ldots,x_{n_2}\}.$$

Continuing this process we find $0 = n_0 \leq n_1 \leq n_2 \leq n_3 \leq \ldots$ and $x_1,x_2,x_3,\ldots \in A$ such that $x_{n_k+1},\ldots,x_{n_{k+1}} \in U_k$ for every $k \in \mathbb{N}$ and

$$A = U_k + \mathrm{co}\,\{x_1,\ldots,x_{n_k}\} \quad (k \geq 1).$$

Then $\lim_{n\to\infty} x_n = 0$ and $A = \overline{\mathrm{co}}\{x_1,x_2,x_3,\ldots\}$. □

We will conclude this part with the following equivalence theorem.

5.2.21 Theorem *Let (A,τ) be a locally convex B_K-module. Then the following assertions are equivalent.*

(α) *(A,τ) is a pure compactoid B_K-module.*

(β) *Every continuous seminorm on A is bounded.*

(γ) *Every continuous seminorm on A has a maximum on A.*

(δ) *If $U_0 \subset U_1 \subset U_2 \subset \ldots$ are open submodules of A such that $\bigcup_{n\in\mathbb{N}} U_n = A$, then there exists an $n \in \mathbb{N}$ such that $U_n = A$.*

For the proof we need the following algebraic lemma.

5.2.22 Lemma *Let M be a B_K-module such that every seminorm on M is bounded. Then M is finitely generated.*

Proof: 1. We first prove the lemma for torsion free B_K-modules. Hence, suppose M is torsion free. Then M is an absolutely convex subset of a K-vector space E, with $E = [M]$ (namely $E = K \otimes_{B_K} M$). E has a vector space base $(e_\iota)_{\iota\in I}$.

Then I is finite. For suppose not. Then there exists an injection $j : \mathbb{N} \to I$. Let

$$D_0 = [(e_\iota)_{\iota\in I\setminus j(\mathbb{N})}] \quad \text{and} \quad D_1 = [(e_\iota)_{\iota\in j(\mathbb{N})}].$$

For each $n \in \mathbb{N}$ there exists a $\lambda_n \in B_K\setminus\{0\}$ such that $\lambda_n e_{j(n)} \in M$.
Define $p : E \to [0,\infty)$ as follows. Let $x \in E$. Let $y \in D_0$ and $z \in D_1$ such

that $x = y + z$. Let $(\mu_n)_{n\in\mathbb{N}} \in K$ such that $z = \sum_{n\in\mathbb{N}} \mu_n e_{j(n)}$, here $\mu_n = 0$ for large n. Then

$$p(x) = \max_{n\in\mathbb{N}} n|\mu_n \lambda_n^{-1}|$$

It is easy to see that p is a (vector space) seminorm on E. Then $p|M$ is a seminorm on M.
For every $n \in \mathbb{N}$ we have that $\lambda_n e_{j(n)} \in M$ and $p|M(\lambda_n e_{j(n)}) = n$. Hence, $p|M$ is not bounded, a contradiction.
Thus I is finite and hence E is finite dimensional. Let $e_1, \ldots, e_n$ be a (vector space) base of E.
By Lemma 4.2.2 M is countably generated. Suppose that M is not finitely generated. Let $\lambda_1, \ldots, \lambda_n \in B_K\backslash\{0\}$ such that $\lambda_i e_i \in M \quad (i = 1, \ldots, n)$. Then $\operatorname{co}\{\lambda_1 e_1, \ldots, \lambda_n e_n\} \neq M$.
Let $f_0 := 0$. As M is countably generated there exist $f_1, f_2, f_3, \ldots \in M$ such that

$$M = \operatorname{co}\{\lambda_1 e_1, \ldots, \lambda_n e_n, f_1, f_2, f_3, \ldots\}$$

and

$$f_m \notin \operatorname{co}\{\lambda_1 e_1, \ldots, \lambda_n e_n, f_1, \ldots, f_{m-1}\} \quad (m \geq 1).$$

We define $p : M \to [0, \infty)$ by

$$p(x) = \min\{m \in \mathbb{N} \mid x \in \operatorname{co}\{\lambda_1 e_1, \ldots, \lambda_n e_n, f_1, \ldots, f_m\}\}$$

. Then

(i) $0 \in \operatorname{co}\{\lambda_1 e_1 \ldots, \lambda_n e_n\}$, hence $p(0) = 0$.

(ii) Let $x, y \in M$. Let $l, m \in \mathbb{N}$ such that $p(x) = l$ and $p(y) = m$. We may assume $m \geq l$. Then $x + y \in \operatorname{co}\{\lambda_1 e_1, \ldots, \lambda_n e_n, f_1, \ldots, f_m\}$ and hence $p(x + y) \leq m = \max(p(x), p(y))$.

(iii) Let $x \in M$ and $\lambda \in B_K$. Let $m \in \mathbb{N}$ be such that $p(x) = m$. Then $x \in \operatorname{co}\{\lambda_1 e_1, \ldots, \lambda_n e_n, f_1, \ldots, f_m\}$ and therefore also $\lambda x \in \operatorname{co}\{\lambda_1 e_1, \ldots, \lambda_n e_n, f_1, \ldots, f_m\}$. Hence $p(\lambda x) \leq m = p(x)$.

(iv) Let $x \in M$ and let $(\mu_k)_{k\in\mathbb{N}} \in B_K$ such that $\mu_k \to 0 \quad (k \to \infty)$. Now $x \in E$, hence there exist $\nu_1, \ldots, \nu_n \in K$ such that $x = \nu_1 e_1 + \cdots + \nu_n e_n$. As $\mu_k \to 0$, we have that $|\mu_k \nu_i| < |\lambda_i|$ for large $k \quad (i = 1, \ldots, n)$. Hence, $\mu_k x \in \operatorname{co}\{\lambda_1 e_1, \ldots, \lambda_n e_n\}$ for large k and therefore $p(\mu_k x) = 0$ for large k, so $p(\mu_k x) \to 0 \quad (k \to \infty)$.

We see that p is a seminorm. For each $m \in \mathbb{N}$ we have that $p(f_m) = m$, which means that p is not bounded, a contradiction.
Thus, M is finitely generated.
2. Now we prove the lemma for torsion B_K-modules, hence suppose M is a torsion B_K-module. Note that $|K|$ is non-trivial in this case.
(i) We first prove that there exists a $\lambda \in B_K\backslash\{0\}$ such that $\lambda M = \{0\}$.
Let $\nu \in B_K$, $0 < |\nu| < 1$. Then $A_i := \{x \in M \mid \nu^i x = 0\}$ is a submodule of M for each $i \in \mathbb{N}$, $A_0 \subset A_1 \subset A_2 \subset \ldots$ and $\bigcup_{i\in\mathbb{N}} A_i = M$. We define $p : M \to [0, \infty)$ by

$$p(x) = \min\{i \in \mathbb{N} \mid x \in A_i\}.$$

It is not hard to see that p is a seminorm. By boundedness of p there exists an $n \in \mathbb{N}$ such that $A_n = M$, that is to say $v^n M = \{0\}$.
(ii) We now prove that there exist an $n \in \mathbb{N}$ and $e_1, \dots, e_n \in M$ such that $M = \mathrm{co}\{e_1, \dots, e_n\} + M^-$.
Let $\pi : M \to M/M^-$ be the canonical map.
Let $(\hat{e}_\iota)_{\iota \in I}$ be a base for the $\mathbf{k}$-vector space M/M^-.
Suppose that I is not finite. Then there exists an injection $j : \mathbb{N} \to I$. We define a seminorm p on M/M^- as follows. Let $x \in M/M^-$. Then there exist a $y \in [(\hat{e}_\iota)_{\iota \in I \setminus j(\mathbb{N})}]$ and $(\mu_n)_{n \in \mathbb{N}} \in \mathbf{k}$ with $\mu_n = 0$ for large n such that $x = y + \sum_{n=0}^{\infty} \mu_n \hat{e}_{j(n)}$. Then $p(x) := \max\{n \mid \mu_n \neq 0\}$.
Now p is not bounded for $p(\hat{e}_{j(n)}) = n \quad (n \in \mathbb{N})$. Then $p \circ \pi$ is an unbounded seminorm on M, a contradiction.
Hence, I is finite. Say $\hat{e}_1, \dots, \hat{e}_n$ is a base for M/M^-. Let $e_1, \dots, e_n \in M$ such that $\pi(e_\iota) = \hat{e}_\iota \quad (i = 1, \dots, n)$. Then $M = \mathrm{co}\{e_1, \dots, e_n\} + M^-$.
(iii) Finally we prove that $M = \mathrm{co}\{e_1, \dots, e_n\}$.
If the valuation on K is discrete, there exists a $\mu \in B_K$ with $0 < |\mu| < 1$ such that $M^- = \mu M$.
If $|K|$ is dense, let $\mu_0, \mu_1, \mu_2, \dots \in B_K \setminus \{0\}$ such that $|\mu_0| < |\mu_1| < |\mu_2| < \dots$ and $\lim_{k \to \infty} |\mu_k| = 1$.
Let $B_m = \mathrm{co}\{e_1, \dots, e_n\} + \mu_m M$. Then B_m is a submodule of M for every $m \in \mathbb{N}$, $B_0 \subset B_1 \subset B_2 \subset \dots$ and $\bigcup B_m = M$.
The seminorm $q : M \to [0, \infty)$ defined by

$$q(x) = \min\{i \in \mathbb{N} \mid x \in B_\iota\}$$

is bounded on M, hence there exists an $m \in \mathbb{N}$ such that $B_m = M$. Then $M = \mathrm{co}\{e_1, \dots, e_n\} + \mu_m M$ and $|\mu_m| < 1$.
Thus in both cases there exists a $\mu \in B_K^- \setminus \{0\}$ such that

$$M = \mathrm{co}\{e_1, \dots, e_n\} + \mu M.$$

Let $x \in M$. Then there exist a $z_1 \in \mathrm{co}\{e_1, \dots, e_n\}$ and a $y_1 \in M$ such that $x = z_1 + \mu y_1$. There exist $w_2 \in \mathrm{co}\{e_1, \dots, e_n\}$ and $y_2 \in M$ such that $y_1 = w_2 + \mu y_2$. Let $z_2 = z_1 + \mu w_2 \in \mathrm{co}\{e_1, \dots, e_n\}$. Then $x = z_2 + \mu^2 y_2$. In this way we find $z_1, z_2, z_3, \dots \in \mathrm{co}\{e_1, \dots, e_n\}$ and $y_1, y_2, y_3, \dots \in M$ such that

$$x = z_k + \mu^k y_k \quad (k \in \mathbb{N}).$$

From (i) we know that there exists a $\lambda \in B_K \setminus \{0\}$ such that $\lambda M = \{0\}$. Let $l \in \mathbb{N}$ such that $|\mu^l| < |\lambda|$. Then $\mu^l y_l = 0$, so $x = z_l \in \mathrm{co}\{e_1, \dots, e_n\}$.
Thus $M = \mathrm{co}\{e_1, \dots, e_n\}$.
3. Finally, let M be an arbitrary B_K-module on which every seminorm is bounded. Let M_t be the torsion part of M. Then M/M_t is torsion free and every seminorm on M/M_t is bounded, for if p is a seminorm on M/M_t then $p \circ \pi$ is a seminorm on M. (Here $\pi : M \to M/M_t$ is the canonical map.) From 1. we obtain that M/M_t is finitely generated. Let $e_1, \dots, e_n \in M$ be such that $\pi(e_1), \dots, \pi(e_n)$ is a minimally generating collection for M/M_t. Then $M = \mathrm{co}\{e_1, \dots, e_n\} + M_t$.
That this is a direct sum is seen in the same way as in the proof of Proposition 2.2.13. It follows that $M_t \sim M/\mathrm{co}\{e_1, \dots, e_n\}$ and thus every seminorm

on M_t is bounded. From 2. we see that M_t is finitely generated, hence so is M. □

Proof of Theorem 5.2.21:
(α) ⇒ (γ): Let p be a continuous seminorm on A. If $p = 0$ we are done. Suppose $p \neq 0$. Let $\delta \in (0, \infty)$ such that $\delta < \sup p$. Let

$$U = \{x \in A \mid p(x) < \delta\}.$$

Then U is an open submodule of A. By Proposition 5.2.17 there exist $x_1, \dots, x_n \in A$ such that $A = U + \mathrm{co}\{x_1, \dots, x_n\}$. We prove that p has its maximum in one of the points $x_1, \dots, x_n$. To this end let $x \in A$ be such that $p(x) > \delta$. There exist a $u \in U$ and $\lambda_1, \dots, \lambda_n \in B_K$ such that $x = u + \sum_{i=1}^{n} \lambda_i x_i$. Then

$$p(x) = p(u + \sum_{i=1}^{n} \lambda_i x_i) \leq \max(p(u), p(x_1), \dots, p(x_n)).$$

Now $\max(p(u), p(x_1), \dots, p(x_n)) = \max(p(x_1), \dots, p(x_n))$, because of the fact that $p(x) > p(u)$. Thus, $p(x) \leq \max(p(x_1), \dots, p(x_n))$.
(γ) ⇒ (δ): Let $U_0 \subset U_1 \subset U_2 \subset \dots$ be open submodules of A such that $\bigcup_{n \in \mathbb{N}} U_n = A$. Define $p : A \to [0, \infty)$ by

$$p(x) = \min\{n \in \mathbb{N} \mid x \in U_n\}.$$

Then p is a seminorm on A (property (iv) is satisfied because U_0 is open and hence absorbing) and p is continuous, for $\mathrm{Ker}\, p$ is open. As p is bounded there exists an $n \in \mathbb{N}$ such that $p < n$. Then $U_n = A$.
(δ) ⇒ (β): Let p be a continuous seminorm on A. Let

$$U_n = \{x \in A \mid p(x) < n\} \quad (n \geq 1).$$

Then each U_n is an open submodule, $U_1 \subset U_2 \subset U_3 \subset \dots$ and $\bigcup_{n \geq 1} U_n = A$. By ($\delta$) there exists an $n \in \mathbb{N}$ such that $U_n = A$. Then $p(x) < n \quad (x \in A)$.
(β) ⇒ (α): Let U be a zero neighbourhood in A. We may assume that U is a submodule of A. Let p be a seminorm on A/U. Then $p \circ \pi$ is a seminorm on A ($\pi : A \to A/U$ is the quotient map) and $p \circ \pi$ is continuous, for $U \subset \mathrm{Ker}\, p \circ \pi$. Hence $p \circ \pi$ is bounded, which implies that also p must be bounded. From Lemma 5.2.22 we obtain that A/U is finitely generated. □

Relative Compactoidity

5.2.23 Definition Let (B, τ) be a locally convex B_K-module and let A be a submodule of B. Then A is called a *compactoid in B* if for every open submodule U of B there exist $x_1, \dots, x_n \in B$ such that $A \subset U + \mathrm{co}\{x_1, \dots, x_n\}$.

From Proposition 5.2.17 it follows that a pure compactoid (A, τ) is compactoid in itself and conversely, a locally convex B_K-module (A, τ) that is compactoid in itself is a pure compactoid.
In the next theorem we prove that compactoidity for a locally convex B_K-module A is the same as that A is compactoid in B for some $B \supset A$. More precisely,

5.2.24 Theorem *Let (A, τ) be a Hausdorff locally convex B_K-module. Then: (A, τ) is a compactoid B_K-module $\Longleftrightarrow$ There exists a (pure compactoid) B_K-module (B, σ), and a homeomorphism j from (A, τ) in (B, σ) such that $j(A)$ is a compactoid in (B, σ).*

Proof: $\Rightarrow$) In Theorem 5.2.5 we have seen that there exist finitely generated discrete torsion B_K-modules $(A_i)_{i \in I}$ such that (A, τ) is topologically embeddable in $\prod_{i \in I} A_i$. Let $B := \prod_{i \in I} A_i$ and let σ be the product topology on B.
Let j be a continuous, open and injective homomorphism from (A, τ) in (B, σ). Then (A, τ) is topologically isomorphic with $(j(A), \sigma | j(A))$.
Now (B, σ) is a pure compactoid B_K-module, since every A_i provided with the discrete topology is a pure compactoid B_K-module.
Let U be an open submodule of B. Then there exist $x_1, \ldots, x_n \in B$ such that $B \subset U + \mathrm{co}\{x_1, \ldots, x_n\}$. Then also $j(A) \subset U + \mathrm{co}\{x_1, \ldots, x_n\}$.
We see that $j(A)$ is a compactoid in (B, σ).
$\Leftarrow$) Let (B, σ) be a locally convex B_K-module and let $j : (A, \tau) \to (B, \sigma)$ be an injective, continuous and open homomorphism from (A, τ) in (B, σ), such that $j(A)$ is compactoid in (B, σ).
Let U be an open submodule of $j(A)$. Let V be an open submodule of B such that $U = V \cap j(A)$. There exist $x_1, \ldots, x_n \in B$ such that $j(A) \subset V + \mathrm{co}\{x_1, \ldots, x_n\}$. Let $\pi : B \to B/V$ be the quotient map. Then $\pi(j(A)) \subset \mathrm{co}\{\pi(x_1), \ldots, \pi(x_n)\}$. Now $\mathrm{co}\{\pi(x_1), \ldots, \pi(x_n)\} \in \mathcal{B}_K$ and hence $\pi(j(A)) \in \mathcal{B}_K$. Let $\rho := \pi \circ i : j(A) \to B/V$, where $i : j(A) \to B$ is the inclusion map. Then $\mathrm{Im}\,\rho = \pi(j(A))$ and $\mathrm{Ker}\,\rho = U$. Hence, $j(A)/U \sim \pi(j(A))$ and therefore also $j(A)/U \in \mathcal{B}_K$.
We see that $(j(A), \sigma | j(A))$ is a compactoid B_K-module and as (A, τ) is topologically isomorphic with $(j(A), \sigma | j(A))$ we obtain that also (A, τ) is a compactoid B_K-module. □

5.2.25 Remark We can also define the notion *locally compactoid in* as follows.
Let (B, τ) be a locally convex B_K-module and let A be a submodule of B. Then A is called a *locally compactoid in* B if for every open submodule U of B there exists a submodule S of finite rank of B such that $A \subset U + S$.

It is not hard to prove that a product of local compactoids in itself is again a local compactoid in itself.
In the same way as we proved Theorem 5.2.24 we can prove the following for a locally convex B_K-module (A, τ)

(A, τ) is a locally compactoid $\Longleftrightarrow$ There exists a locally convex B_K-module (B, σ) (that is a locally compactoid in itself) and a homeomorphism j from A in B such that $j(A)$ is a locally compactoid in B.

5.3 c-compactness

In this section we assume that K is spherically complete.

5.3.1 Definition A locally convex Hausdorff B_K-module (A, τ) is called *c-compact* if the following assertion is true.
If C is a collection of non-empty closed convex subsets of A with the finite intersection property, then $\bigcap C \neq \emptyset$.

5.3.2 Remark This definition of c-compactness is literally taken over from K-vector space theory, so the following will not be amazing.
Let (E, τ) be a locally convex K-vector space and let A be an absolutely convex subset of E. Then
$(A, \tau|A)$ is a c-compact B_K-module $\iff$ A is a c-compact subset of E.

5.3.3 Remark For a torsion module in $\mathcal{F}_K$ and for modules over trivial valued B_K the discrete topology is a locally convex topology. For these B_K-modules linear compactness (see Definition 2.3.10) is the same as c-compactness with respect to the discrete topology. Furthermore, every B_K-module in $\mathcal{F}_K$ equipped with any locally convex topology is c-compact since it is linearly compact (see Theorem 2.3.22).

5.3.4 Definition Let (A, τ) be a locally convex B_K-module, let $\mathcal{F}$ be a filter on A and let $x \in A$. We say $\mathcal{F}$ *converges to* x if $U \in \mathcal{F}$ for every neighbourhood U of x. We call x *adherent on* $\mathcal{F}$ if $x \in \overline{A}$ for every $A \in \mathcal{F}$.

For the following lemma we recall that a filter is called convex if it has a convex base.

5.3.5 Lemma *Let (A, τ) be a locally convex B_K-module and let $\mathcal{M}$ be a maximal convex filter on A. Let $x \in A$. Then:*
$\mathcal{M}$ converges to x $\iff$ x is adherent on $\mathcal{M}$.

Proof: $\Rightarrow$) Let $C \in \mathcal{M}$. For every neighbourhood U of x we have $U \in \mathcal{M}$ and hence $C \cap U \neq \emptyset$. Thus, $x \in \overline{C}$. This is true for every $C \in \mathcal{M}$ and hence x is adherent on $\mathcal{M}$.
$\Leftarrow$) Let U be a neighbourhood of x in A. Then there exists a convex neighbourhood V of x such that $V \subset U$. We have $x \in \overline{C}$ for every $C \in \mathcal{M}$, hence $V \cap C \neq \emptyset$ for all $C \in \mathcal{M}$. As $\mathcal{M}$ is a maximal convex filter it follows that $V \in \mathcal{M}$, which implies that also $U \in \mathcal{M}$. We see that $\mathcal{M}$ converges to x. □

5.3.6 Theorem *Let (A, τ) be a locally convex B_K-module. Then the following assertions are equivalent.*

(i) *(A, τ) is c-compact.*

(ii) *Every convex filter on A has an adherent point $x \in A$.*

(iii) *Every maximal convex filter on A converges.*

Proof: (ii) $\Leftrightarrow$ (iii) follows from Lemma 5.3.5 and from the fact that every convex filter can be extended to a maximal convex filter.
(i)$\Rightarrow$(ii): Let (A, τ) be c-compact. Let $\mathcal{F}$ be a convex filter on A. Let $\mathcal{B}$ be a convex base of $\mathcal{F}$. Then $\overline{\mathcal{B}} := \{\overline{B} \mid B \in \mathcal{B}\}$ a collection non-empty closed convex subsets of A with the finite intersection property. Since (A, τ) is c-compact there exists an $x \in A$ such that $x \in \overline{B}$ for every $B \in \mathcal{B}$. Then also $x \in \overline{D}$ for every $D \in \mathcal{F}$. Thus x is adherent on $\mathcal{F}$.
(ii)$\Rightarrow$(i): Let C be a collection of non-empty closed convex subsets of A with the finite intersection property. Let $\mathcal{G}$ be the filter generated by C. Then there exists an $x \in A$ such that $x \in \overline{D}$ for every $D \in \mathcal{G}$. Then, of course $x \in \overline{C} = C$ for every $C \in C$. □

5.3.7 Theorem *A c-compact B_K-module is complete.*

Proof: (See also the proof of Proposition 4.1.8.) Let $(x_\alpha)_{\alpha \in I}$ be a Cauchy net in A. For $\alpha \in I$ let

$$C_\alpha = \{\sum_{\beta \succ \alpha} \lambda_\beta x_\beta \mid \lambda_\beta \in B_K,\ \lambda_\beta \neq 0 \text{ for finitely many } \beta \text{ and } \sum_{\beta \succ \alpha} \lambda_\beta = 1\}$$

be the convex hull of $\{x_\beta \mid \beta \succ \alpha\}$. Then each C_α is convex. Let C be the collection $(C_\alpha)_{\alpha \in I}$ and let $\mathcal{F}$ be the filter generated by C. By Theorem 5.3.6 there exists an $x \in A$ such that x is adherent on $\mathcal{F}$.
Suppose not $x_\alpha \to x$. Then there exists a convex neighbourhood U of x such that for every $\alpha \in I$ there exists a $\beta \succ \alpha$ such that $x_\beta \notin U$. (Here $\succ$ is the partial order on I.)
Let $\alpha \in I$ such that $x_\gamma - x_\beta \in U - \{x\}$ for every $\beta, \gamma \succ \alpha$. Let $\beta \succ \alpha$ such that $x_\beta \notin U$. Let $V = U - \{x\} + \{x_\beta\}$.
V is convex and open and hence also closed. Furthermore, $x_\gamma \in V$ for all $\gamma \succ \beta$, hence $\overline{C_\beta} \subset V$, and therefore also $x \in V$. Then $x = u - x + x_\beta$ for some $u \in U$. Then $x_\beta = x + x - u$ and $x + x - u \in U$, for U is convex. This is in conflict to the choice of β. Hence, $x_\alpha \to x$.
We see that (A, τ) is complete. □

5.3.8 Theorem *Let (A, τ) be c-compact. Let B be a closed submodule of A. Then $(B, \tau|B)$ is also c-compact.*

Proof: Let C be a collection of non-empty closed convex subsets of B with the finite intersection property. Then every $C \in C$ is also closed in A. Thus, there exists an $x \in A$ such that $x \in \bigcap C$. And $\bigcap C \subset B$, hence $x \in B$. □

5.3.9 Theorem *Let (A, τ) be a c-compact B_K-module. Let (B, σ) be a locally convex B_K-module and let $\varphi : A \to B$ be a continuous surjective homomorphism. Then (B, σ) is c-compact.*

Proof: Let C be a collection of non-empty closed convex subsets of B with the finite intersection property. Let $\mathcal{D}$ be the collection $(\varphi^{-1}(C))_{C \in C}$. Since every $C \in C$ is non-empty, convex and closed and φ is continuous we have that every $D \in \mathcal{D}$ non-empty, convex and closed.

Let $D_1, \dots, D_n \in \mathcal{D}$. Let $C_1, \dots C_n \in C$ with $D_\iota = \varphi^{-1}(C_\iota) \quad (i = 1, \dots n)$. Then $\bigcap_{\iota=1}^{n} C_\iota \neq \emptyset$ and hence

$$\bigcap_{\iota=1}^{n} D_\iota = \bigcap_{\iota=1}^{n} \varphi^{-1}(C_\iota) = \varphi^{-1}(\bigcap_{\iota=1}^{n} C_\iota) \neq \emptyset.$$

Thus, $\mathcal{D}$ has the finite intersection property and as (A, τ) is c-compact there exists an $x \in A$ with $x \in \bigcap \mathcal{D}$. Then $\varphi(x) \in \bigcap C$.
We see that (B, σ) is c-compact. □

5.3.10 Proposition *Let (A, τ) be a c-compact B_K-module and let φ be a continuous homomorphism from (A, τ) to a Hausdorff locally convex B_K-module (B, σ). Let C be a non-empty convex closed subset of A. Then $\varphi(C)$ is closed in B.*

Proof: Let $x \in C$. Let $T_{-x} : A \to A$ and $T_{\varphi(x)} : B \to B$ be the translation maps as defined in Proposition 3.1.6. Then $T_{-x}(C)$ is a closed submodule of A and thus $(T_{-x}(C), \tau | T_{-x}(C))$ is c-compact. Then, by Theorem 5.3.9, $(\varphi(T_{-x}(C)), \sigma | \varphi(T_{-x}(C)))$ is c-compact and hence, by Theorem 5.3.7, complete. As (B, σ) is Hausdorff we obtain that $\varphi(T_{-x}(C))$ is closed. Then $\varphi(C) = T_{\varphi(x)}(\varphi(T_{-x}(C)))$ is also closed. □

5.3.11 Corollary *Let C be a non-empty closed convex subset of a c-compact B_K-module (A, τ). Let $\lambda \in B_K$. Then λC is closed in A.*

Proof: The map $M_\lambda : A \to A$ defined by $M_\lambda(x) = \lambda x \quad (x \in A)$ is continuous and $T_\lambda(C) = \lambda C$. Now apply Proposition 5.3.10. □

5.3.12 Lemma *Let (A, τ) be a locally convex B_K-module and let $\mathcal{B}$ be a base of a maximal convex filter $\mathcal{M}$ on A. Let φ be a continuous surjective homomorphism from (A, τ) to a locally convex B_K-module (B, σ). Then $\varphi(\mathcal{B})$ is a convex base for a maximal convex filter on B.*

Proof: It is clear that $\varphi(\mathcal{B})$ is a base for a convex filter $\mathcal{F}$ on B. We show that $\mathcal{F}$ is a maximal convex filter.
Let C be a convex subset of B. Then $\varphi^{-1}(C)$ is a convex subset of A.
If $\varphi^{-1}(C) \in \mathcal{M}$ then there exists a $B \in \mathcal{B}$ such that $B \subset \varphi^{-1}(C)$. Then $\varphi(B) \subset C$ and hence $C \in \mathcal{F}$.
If $\varphi^{-1}(C) \notin \mathcal{M}$ there exists a $B \in \mathcal{B}$ such that $B \cap \varphi^{-1}(C) = \emptyset$. Then also $\varphi(B) \cap C = \emptyset$.
We see that for every convex subset C of B either $C \in \mathcal{F}$ or $C \cap D = \emptyset$ for some $D \in \mathcal{F}$. Hence, by Proposition 2.3.8, $\mathcal{F}$ is a maximal convex filter on B. □

5.3.13 Theorem *Let I be an index set and for every $i \in I$ let (A_ι, τ_ι) be a c-compact B_K-module. Then $A := \prod_{\iota \in I} A_\iota$, provided with the product topology is also c-compact.*

Proof: Let $\mathcal{M}$ be a maximal convex filter on A. We prove that $\mathcal{M}$ converges. From Lemma 5.3.12 we see that $P_\iota(\mathcal{M})$ is a base for a maximal convex filter $\mathcal{M}_\iota$ on A_ι for each $\iota \in I$. From Theorem 5.3.6 we obtain that for each $\iota \in I$ there exists an $x_\iota \in A_\iota$ such that $\mathcal{M}_\iota$ converges to x_ι.
Let $x \in A$ such that $P_\iota(x) = x_\iota \quad (\iota \in I)$. We prove that $\mathcal{M}$ converges to x.
Let U be a neighbourhood of x in A. Then there exist $n \in \mathbb{N}$, $\iota_1, \dots, \iota_n \in I$ and $U_1, \dots, U_n$ with $U_j \subset A_{\iota_j}$ is a neighbourhood of x_{ι_j} for $j \in \{1, \dots, n\}$ such that $\bigcap_{j=1}^n P_{\iota_j}^{-1}(U_j) \subset U$.
Let $j \in \{1, \dots, n\}$. Then $U_j \in \mathcal{M}_j$, for $\mathcal{M}_j$ converges to x_j. Hence there exists a $V_j \in P_{\iota_j}(\mathcal{M})$ such that $V_j \subset U_j$. Now $P_{\iota_j}^{-1}(V_j) \in \mathcal{M}$, for $V_j = P_{\iota_j}(C_j)$ for some $C_j \in \mathcal{M}$. Then also $P_{\iota_j}^{-1}(U_j)$ in $\mathcal{M}$.
This is true for all $j \in \{1, \dots, n\}$, and therefore also $\bigcap_{\iota=1}^n P_{\iota_j}^{-1}(U_j) \in \mathcal{M}$. And hence also $U \in \mathcal{M}$.
From Theorem 5.3.6 we see that A provided with the product topology is c-compact. □

The next theorem connects c-compactness with local compactoidity.

5.3.14 Theorem *Let (A, τ) be a Hausdorff locally convex B_K-module. Then: (A, τ) is locally compactoid and complete $\iff$ (A, τ) is c-compact.*

Proof: $\Rightarrow$) As (A, τ) is complete, (A, τ) is topologically isomorphic to a closed submodule of $\prod_{U \in C} A/U$, where C is the collection of all open submodules of A. Now A/U is a torsion module in $\mathcal{F}_K$ for every $U \in C$. In Remark 5.3.3 we have seen that then each A/U is c-compact with respect to the discrete topology. By Theorem 5.3.13 also $\prod_{U \in C} A/U$ is c-compact. Finally, Theorem 5.3.8 implies that also (A, τ) is c-compact.
$\Leftarrow$) (A, τ) is c-compact and hence complete. Furthermore, for every open submodule U of A the quotient A/U is c-compact with respect to the discrete topology. By using Theorem 2.3.22 we obtain $A/U \in \mathcal{F}_K$. We see that (A, τ) is locally compactoid. □

5.3.15 Remark As an easy consequence of Proposition 5.2.3 we obtain the following for a locally convex B_K-module (A, τ).
(A, τ) is bounded and c-compact $\iff$ (A, τ) is a complete compactoid.

The following lemma resembles Proposition 2.3.18. We need it for the proof of Theorem 5.3.17 and Theorem 5.3.18.

5.3.16 Lemma *Let (A, τ) be a c-compact B_K-module and let (B, σ) be a Hausdorff locally convex B_K-module. Let $\varphi : A \to B$ be a continuous surjective homomorphism. Let C be an absolutely convex subset of K and let $\rho : C \to B$ be a homomorphism. Then there exists a homomorphism $\vartheta : C \to A$ such that* *the following diagram commutes.*

$$\begin{tikzcd} A \arrow[rr, "\varphi"] & & B \\ & C \arrow[ul, "\vartheta"] \arrow[ur, "\rho"'] & \end{tikzcd}$$

Proof: case 1. There exists a $\mu \in K$ such that $C = \operatorname{co}\{\mu\}$. Let $x \in A$ such that $\varphi(x) = \rho(\mu)$. We define $\vartheta : C \to A$ by $\vartheta(\lambda) = \frac{\lambda}{\mu}x \quad (\lambda \in C)$. Obviously ϑ satisfies the requirements (see also **case 1.** of the proof of Proposition 2.3.18).

case 2. There does not exist a $\mu \in K$ such that $C = \operatorname{co}\{\mu\}$. Then there exist $\lambda_1, \lambda_2, \lambda_3, \ldots \in K$ with $|\lambda_1| < |\lambda_2| < |\lambda_3| < \ldots$ such that

$$C = \operatorname{co}\{\lambda_1, \lambda_2, \lambda_3, \ldots\}.$$

In this case the proof is exactly the same as in **case 2.** of Proposition 2.3.18. The only thing we have to show is that for every $m, n \in \mathbb{N}$ with $m \le n$ the set $\frac{\lambda_m}{\lambda_n}\varphi^{-1}(\rho(\lambda_n))$ is closed. In fact, let $m, n \in \mathbb{N}$ with $m \le n$. The singleton set $\{\rho(\lambda_n)\}$ is closed and hence $\varphi^{-1}(\rho(\lambda_n))$ is a closed subset of A. Then also $\frac{\lambda_m}{\lambda_n}\varphi^{-1}(\rho(\lambda_n))$ is closed in A (see Corollary 5.3.11). □

5.3.17 Theorem *Let (A, τ) be a c-compact B_K-module. Let U be an open submodule of A. Then there exist $n \in \mathbb{N}$ and submodules $F_1, F_2, \ldots, F_n$ of rank ≤ 1 of A such that $A = U + F_1 + F_2 + \cdots + F_n$.*

Proof: Let $\pi : A \to A/U$ be the quotient map. (A, τ) is c-compact and hence, by Theorem 5.3.14, a locally compactoid B_K-module. Thus $A/U \in \mathcal{F}_K$. Let $n = \operatorname{rank} A/U$. By using Theorem 2.2.42 we find submodules $B_1, \ldots, B_n$ of rank 1 of A/U such that $A/U = B_1 + \ldots + B_n$.
Let $C_1, \ldots, C_n$ be absolutely convex subsets of K and for every $i \in \{1, \ldots, n\}$ let $\rho_i : C_i \to B_i$ be a surjective homomorphism. From the previous lemma we obtain that for every $i \in \{1, \ldots, n\}$ there exist a map $\vartheta_i : C_i \to A$ such that $\pi \circ \vartheta_i = \rho_i$. Then $\vartheta_1(C_1), \ldots, \vartheta_n(C_n)$ are rank ≤ 1 submodules of A and $A = U + \vartheta_1(C_1) + \cdots + \vartheta_n(C_n)$. □

As already announced in section 2.4 we prove the following theorem about edge completeness.

5.3.18 Theorem *Let (A, τ) be an edge complete, c-compact B_K-module. Let (B, σ) be a Hausdorff locally convex B_K-module and let $T : (A, \tau) \to (B, \sigma)$ be a continuous, surjective homomorphism. Then (B, σ) is edge complete and c-compact.*

Proof: That (B, σ) is c-compact follows from Theorem 5.3.9. We prove that B is edge complete. To this end, let $\rho : B_K^- \to B$ be a homomorphism. By Lemma

5.3.16 there exists a homomorphism $\vartheta : B_K^- \to A$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[rr, "\varphi"] & & B \\ & B_K^- \arrow[ul, "\vartheta"] \arrow[ur, "\rho"'] & \end{tikzcd}$$

Since A is edge complete there exists a homomorphism $\tilde{\vartheta} : B_K \to A$ that is an extension of ϑ. Then $\varphi \circ \tilde{\vartheta}$ is a homomorphism $B_K \to B$ that is an extension of ρ. □

5.3.19 Remark The following is in general not true.
Let (A, τ) be a c-compact B_K-module and let (B, σ) be a Hausdorff locally convex B_K-module. Let $\varphi : A \to B$ be a continuous homomorphism. Let C be a closed submodule of A that is edged in A. Then $T(C)$ is (c-compact and) edged in B.
For example, $C = A = B = B_K / B_K^-$, provided with the discrete topology and $\varphi = 0$.

We conclude this section with some observations on the on the plus topology on c-compact B_K-modules.

5.3.20 Remark Let the valuation on K be dense. Let (A, τ) be a c-compact B_K-module. Then (A, τ^+) need not be c-compact.
For example, Let $A = (B_K / B_K^-)^{\mathrm{N}}$ and let τ be the product topology on A induced by the discrete topology on B_K / B_K^-. By Remark 5.3.3, B_K / B_K^- is c-compact with respect to the discrete topology. From Theorem 5.3.13 we obtain that then also A is c-compact. In the remarks that precede Proposition 4.4.32 we have seen that τ^+ is discrete. It is not hard to see that $A \notin \mathcal{F}_K$ and hence, by Theorem 2.3.22, A is not linearly compact. This implies (A, τ^+) is not c-compact.

On the metrizability of the plus topology on torsion free c-compact modules we have the following theorem (compare Corollary 4.4.54).

5.3.21 Theorem *Let the valuation on K be dense. Let (A, τ) be a torsion free, bounded c-compact B_K-module. Then: τ^+ is metrizable $\iff A \in \mathcal{B}_K$.*

Proof: $\Rightarrow$) By Proposition 4.4.32 there exists a $\lambda \in B_K^-$ such that $\overline{\lambda A}$ is τ^+-open. (Here $\overline{\lambda A}$ is the closure of λA with respect to τ.) From Corollary 5.3.11 we obtain that λA is closed with respect to τ. Then $\overline{\lambda A} = \lambda A$ and hence λA is τ^+-open.
Let $\mu \in B_K$ such that $|\lambda| < |\mu| < 1$. Then $\mu A = (\mu\lambda^{-1})\lambda A$, for A is torsion free. Now $|\mu\lambda^{-1}| > 1$ and hence, by definition of τ^+, μA is τ-open.
By Theorem 5.3.17 there exists a submodule S of A with $S \in \mathcal{F}_K$ such that $A = S + \mu A$.

Let $x \in A$. Then there exists a $y \in S$ and a $z \in A$ such that $x = y + \mu z$. There exist $y' \in S$ and $z' \in A$ such that $z = y' + \mu z'$. Then

$$x = y + \mu z = y + \mu(y' + \mu z') = (y + \mu y') + \mu^2 z' \in S + \mu^2 A.$$

Continuing this process we see that $x \in S + \mu^n A$ for every $n \in \mathbb{N}$.
Let U be an open submodule of A. As A is bounded there exists an $m \in \mathbb{N}$ such that $\mu^m A \subset U$. Then $A \subset S + \mu^m A \subset S + U$.
We obtain that $A \subset S + U$ for every open submodule U of A and this implies that $A \subset \overline{S}$.
By Theorem 2.3.22 S is linearly compact, thus $(S, \tau|S)$ is c-compact and hence complete. As (A, τ) is Hausdorff it follows that S is closed in A and hence $A = \overline{S} = S \in \mathcal{F}_K$. Since A is bounded we obtain $A \in \mathcal{B}_K$.
$\Leftarrow$) By Proposition 4.1.4 there exists only one Hausdorff locally convex topology on A and this topology is metrizable. □

5.4 Almost-Openness of Surjections

In this section we no longer assume that K is spherically complete, but instead we assume that the valuation on K is dense.

We will prove here that a continuous homomorphism from a metrizable locally compactoid B_K-module (A, τ) to a locally convex Hausdorff B_K-module (B, σ) is almost open. More precisely, the homomorphism is open if we replace σ on B by σ^+.
The main part is also proved in [27] for compactoids instead of locally compactoids. The definition of a compactoid in [27] is slightly different from the definition given in this chapter. Moreover, in [27] no plus- and minus topology are involved. So we set up the theory from scratch.

5.4.1 Lemma *Let (A, τ) be a locally convex B_K-module. Let B be a closed submodule of A and let F be a submodule of rank 1 of A. Let $C \subset K$ be absolutely convex and let $\varphi : C \to F$ be a surjective homomorphism. Let $\lambda \in B_K^-$.*
Let $(b_\alpha)_{\alpha \in I}$ be a net in B and let $(c_\alpha)_{\alpha \in I}$ be a net in C such that

$$x_\alpha := b_\alpha + \varphi(c_\alpha) \xrightarrow{\tau} 0.$$

Then $c_\alpha \to 0$ or $\lambda x_\alpha \in B$ for large α. The latter occurs if B absorbs all elements of F.

Proof: Suppose not $c_\alpha \to 0$. Then there exists a $\nu \in C$, $|\nu| > 0$ and a cofinal subset J of I such that $|c_\beta| \geq |\nu|$ for all $\beta \in J$. Now $x_\beta = b_\beta + \varphi(c_\beta) \to 0$. As $\nu c_\beta^{-1} \in B_K$ for $\beta \in J$ we obtain that $\nu c_\beta^{-1} x_\beta = \nu c_\beta^{-1} b_\beta + \varphi(\nu) \to 0$. And hence $-\nu c_\beta^{-1} b_\beta \to \varphi(\nu)$. Then $\varphi(\nu) \in B$ for B is closed.
We see that $\varphi(\nu) \in B$ for every $\nu \in C$ with $|\nu| < \limsup_{\alpha \in I} |c_\alpha|$.
Now $|c_\alpha| < |\lambda|^{-1} \limsup_{\beta \in I} |c_\beta|$ for large α, hence $|\lambda c_\alpha| < \limsup_{\beta \in I} |c_\beta|$ which implies that $\lambda\varphi(c_\alpha) = \varphi(\lambda c_\alpha) \in B$ for large α.
Then $\lambda x_\alpha = \lambda b_\alpha + \lambda\varphi(c_\alpha) \in B + B = B$ for large α.

Suppose that B absorbs all elements of F. If not $c_\alpha \to 0$ then $\lambda x_\alpha \in B$ for large α as we have already seen.
Suppose $c_\alpha \to 0$. Then there exists a $\mu \in C \setminus \{0\}$ such that $|c_\alpha| < |\mu|$ for large α. Then $\frac{c_\alpha}{\mu} \in B_K$ for large α and $\frac{c_\alpha}{\mu} \to 0$. As B absorbs $\varphi(\mu)$ we obtain that $\varphi(c_\alpha) = \frac{c_\alpha}{\mu}\varphi(\mu) \in B$ and hence $x_\alpha \in B$ for large α. □

We need the following proposition for the proof of the next lemma.

5.4.2 Proposition *Let (A, τ) be a locally convex B_K-module. Let B be a submodule of A. Then $\overline{B^e} \subset \overline{B}^e$. Furthermore, B^e is closed $\iff$ $\overline{B} \subset B^e$ $\iff$ $\lambda\overline{B} \subset B$ for every $\lambda \in B_K^-$.*

Proof: Suppose $x \in \overline{B^e}$. Then there exists a net $(x_\alpha)_{\alpha\in I}$ in B^e such that $x_\alpha \to x$. Then for every $\lambda \in B_K^-$ we have that $(\lambda x_\alpha)_{\alpha\in I}$ is a net in B and $\lambda x_\alpha \to \lambda x$. Hence $\lambda x \in \overline{B}$.
We obtain $x \in \overline{B}^e$.
Furthermore, suppose B^e is closed. Let $x \in \overline{B}$. Then $x \in \overline{B^e} = B^e$. Hence, $\overline{B} \subset B^e$.
Suppose $\overline{B} \subset B^e$. Let $\lambda \in B_K^-$. Let $x \in \overline{B}$. Then $x \in B^e$ and hence $\lambda x \in B$. We see $\lambda\overline{B} \subset B$.
Suppose $\lambda\overline{B} \subset B$ for every $\lambda \in B_K^-$. Let $(x_\alpha)_{\alpha\in I}$ be a net in B^e and $x \in A$ such that $x_\alpha \to x$. Let $\lambda \in B_K^-$. There exists a $\mu \in B_K^-$ such that $|\mu|^2 \ge |\lambda|$ (recall that we assume the valuation to be dense). Then $(\mu x_\alpha)_{\alpha\in I}$ is a net in B and $\mu x_\alpha \to \mu x$. Then $\mu x \in \overline{B}$ and as $\mu\overline{B} \subset B$ we obtain that $\lambda x \in B$.
We see that $\lambda x \in B$ for every $\lambda \in B_K^-$ and hence $x \in B^e$. Thus, B^e is closed.
□

5.4.3 Lemma *Let (A, τ) be a locally convex B_K-module and let B be a closed submodule of A. Let F be a submodule of rank 1 of A. Then $(B+F)^e$ is closed.*

Proof: By Proposition 5.4.2 it suffices to prove that $\overline{B+F} \subset (B+F)^e$.
Let C be an absolutely convex subset of K and let $\varphi : C \to F$ be a surjective homomorphism.
Let $x \in \overline{B+F}$. Let $(x_\alpha)_{\alpha\in I}$ be a net in $B+F$ such that $x_\alpha \to x$. Let $(b_\alpha)_{\alpha\in I}$ be a net in B and $(c_\alpha)_{\alpha\in I}$ be a net in C such that $x_\alpha = b_\alpha + \varphi(c_\alpha)$ $(\alpha \in I)$.
The net $(x_\alpha - x_\beta)_{(\alpha,\beta)\in I\times I}$ tends to 0. Let $\lambda \in B_K^-$. From Lemma 5.4.1 we obtain that either $c_\alpha - c_\beta \to 0$ or $\lambda(x_\alpha - x_\beta) \in B$ eventually.
In the first case the net $(c_\alpha)_{\alpha\in I}$ converges in C for K is complete and C is closed in K. Let $c \in C$ such that $c_\alpha \to c$. Then $b_\alpha = x_\alpha - \varphi(c_\alpha) \to x - \varphi(c)$ and as B is closed we obtain that $x - \varphi(c) \in b$. Then

$$x = (x - \varphi(c)) + \varphi(c) \in B + F \subset (B+F)^e.$$

In the second case let $\gamma \in I$ such that $\lambda(x_\alpha - x_\gamma) \in B$ for every $\alpha \succ \gamma$. Then $\lambda\varphi(c_\alpha - c_\gamma) = \lambda(x_\alpha - x_\gamma) - \lambda(b_\alpha - b_\gamma) \in B - B = B$ and hence $\lambda x_\alpha = \lambda x_\gamma + \lambda(b_\alpha - b_\gamma) + \lambda\varphi(c_\alpha - c_\gamma) \in \lambda x_\gamma + B$ for $\alpha \succ \gamma$. Hence

$$\lambda x \in \overline{\lambda x_\gamma + B} = \lambda x_\gamma + B \subset (\lambda B + \lambda F) + B \subset B + F.$$

We see that $\lambda x \in B + F$ for all $\lambda \in B_K^-$ and hence $x \in (B+F)^e$. □

5.4.4 Lemma *Let (A,τ) be a locally convex B_K-module and let B be a closed submodule of A. Let $F_1,\dots,F_n$ be submodules of rank 1 of A. Then $(B+(F_1+\cdots+F_n))^e$ is closed.*

Proof: By induction. The case $n=1$ is Lemma 5.4.3.
Let $n\in\mathbb{N}$ be such that $(B+(F_1+\cdots+F_n))^e$ is closed for all submodules $F_1,\dots,F_n$ of A with rank $F_i=1$ $(i=1,\dots,n)$. Let $F_1,\dots,F_{n+1}$ submodules of rank 1 of A. By induction $(B+(F_1+\cdots+F_n))^e$ is closed. From Lemma 5.4.3 we obtain that $((B+(F_1+\cdots+F_n))^e+F_{n+1})^e$ is closed and hence, by Proposition 5.4.2,

$$\overline{B+(F_1+\cdots+F_{n+1})}\subset\overline{(B+(F_1+\ldots+F_n))^e+F_{n+1}}\subset$$
$$((B+(F_1+\cdots+F_n))^e+F_{n+1})^e.$$

From Proposition 2.4.5 we obtain that

$$((B+(F_1+\cdots+F_n))^e+F_{n+1})^e=(B+(F_1+\cdots+F_n+F_{n+1}))^e.$$

By using Proposition 5.4.2 again we obtain that $(B+(F_1+\cdots+F_n+F_{n+1}))^e$ is closed. □

5.4.5 Lemma *Let (A,τ) be a locally convex B_K-module and let B be a closed submodule of A. Let $F_1,\dots,F_n$ be submodules of rank 1 of A, such that B absorbs every element of $F_1+\cdots+F_n$. If $\lambda\in B_K^-$ and $(x_\alpha)_{\alpha\in I}$ is a net in $B+(F_1+\cdots+F_n)$, with $x_\alpha\to 0$, then $\lambda x_\alpha\in B$ for large α.*

Proof: Let $\mu_1,\dots,\mu_n\in B_K^-$ such that $1>|\mu_1\mu_2\cdots\mu_n|^2>|\lambda|$. Clearly, $x_\alpha\in\overline{B+(F_1+\cdots+F_{n-1})}+F_n$. Then Lemma 5.4.1 implies that

$$\mu_n x_\alpha\in\overline{B+(F_1+\cdots+F_{n-1})}\text{ for large }\alpha.$$

As $(B+(F_1+\cdots+F_{n-1}))^e$ is closed (by Lemma 5.4.4) we obtain by Proposition 2.4.5 that

$$\overline{B+(F_1+\cdots+F_{n-1})}\subset(B+(F_1+\cdots+F_{n-1}))^e$$

and hence $\mu_n x_\alpha\in(B+(F_1+\cdots+F_{n-1}))^e$ for large α. Then

$$\mu_n^2 x_\alpha\in B+(F_1+\cdots+F_{n-1})\subset\overline{B+(F_1+\cdots+F_{n-2})}+F_{n-1}$$

for large α. By repeating the above argument we find for even larger α that

$$\mu_{n-1}^2\mu_n^2 x_\alpha\in B+(F_1+\cdots+F_{n-2}).$$

Continuing this way we obtain

$$\mu_1^2\mu_2^2\cdots\mu_n^2 x_\alpha\in B\text{ for very large }\alpha.$$

And as $|\mu_1\mu_2\cdots\mu_n|^2>|\lambda|$ we obtain that $\lambda x_\alpha\in B$ eventually. □

5.4.6 Theorem *Let (A,τ) and (B,σ) be Hausdorff locally convex B_K-modules. Let $T : A \to B$ be a continuous homomorphism. Let C be a complete metrizable locally compactoid submodule of A and let D be a closed submodule of B. Then $(TC+D)^e$ is closed.*

Proof: Let $\lambda \in B_K^-$. We prove that $\lambda\overline{TC+D} \subset TC+D$ (see Proposition 5.4.2). Let $\mu_0,\mu_1,\mu_2,\mu_3,\ldots \in B_K^-$ such that $|\mu_0| > |\mu_1| > |\mu_2| > \cdots > |\lambda|$. By using Theorem 5.1.21 we obtain that there exist submodules $F_1,F_2,F_3,\ldots$ of rank ≤ 1 of C with $\lim_{n\to\infty} F_n = \{0\}$ such that $\mu_0 C \subset \overline{\mathrm{co}}\{F_1,F_2,F_3,\ldots\} \subset C$.
Then $\lim_{n\to\infty} TF_n = \{0\}$ and

$$\mu_0 TC = T(\mu_0 C) \subset T\left(\overline{\mathrm{co}}\{F_1,F_2,F_3,\ldots\}\right) \subset TC.$$

Let $C_n = \overline{\mathrm{co}}\{F_n,F_{n+1},F_{n+2},\ldots\}$ $(n \geq 1)$. Then $\lim_{n\to\infty} C_n = 0$. Now

$$\mu_0\overline{TC+D} \subset \overline{\mu_0(TC+D)} = \overline{\mu_0 TC + \mu_0 D} \subset \overline{TC_1+D}.$$

From Lemma 5.4.3 we obtain that $(TF_1 + \overline{TC_2+D})^e$ is closed and hence

$$\overline{TF_1 + \overline{TC_2+D}} \subset (TF_1 + \overline{TC_2+D})^e.$$

This implies that

$$\frac{\mu_1}{\mu_0}\overline{TC_1+D} = \frac{\mu_1}{\mu_0}\overline{TF_1+TC_2+D} \subset \frac{\mu_1}{\mu_0}\overline{TF_1+\overline{TC_2+D}} \subset TF_1 + \overline{TC_2+D}.$$

And in the same way

$$\frac{\mu_n}{\mu_{n-1}}\overline{TC_n+D} \subset TF_n + \overline{TC_{n+1}+D} \quad (n \geq 1).$$

Now let $x \in \lambda\overline{TC+D}$. Then

$$x \in \frac{\lambda}{\mu_0}\mu_0\overline{TC+D} \subset \frac{\lambda}{\mu_0}\overline{TC_1+D} = \frac{\lambda}{\mu_1}\left(\frac{\mu_1}{\mu_0}\overline{TC_1+D}\right) \subset$$
$$\frac{\lambda}{\mu_1}(TF_1 + \overline{TC_2+D}) \subset TF_1 + \frac{\lambda}{\mu_1}\overline{TC_2+D}.$$

Hence there exists a $y_1 \in F_1$ and a $z_1 \in \frac{\lambda}{\mu_1}\overline{TC_2+D}$ such that $x = Ty_1 + z_1$. Now $z_1 \in \frac{\lambda}{\mu_1}\overline{TC_2+D} \subset TF_2 + \frac{\lambda}{\mu_2}\overline{TC_3+D}$ and hence there exist a $y_2 \in F_2$ and a $z_2 \in \frac{\lambda}{\mu_2}\overline{TC_3+D}$ such that $z_1 = Ty_2 + z_2$. Then $x = (Ty_1+Ty_2)+z_2$. Continuing this process we find $y_1,y_2,y_3,\ldots \in C$ with $y_n \in F_n$ $(n \geq 1)$ and $z_1,z_2,z_3,\ldots \in B$ with $z_n \in \frac{\lambda}{\mu_{n-1}}\overline{TC_n+D}$ $(n \geq 1)$ such that $x = \sum_{i=1}^n Ty_i + z_n$ $(n \geq 1)$. Now $\lim_{n\to\infty} y_n = 0$ as $y_n \in F_n$ $(n \geq 1)$ and $\lim_{n\to\infty} F_n = \{0\}$. As C is complete $\sum_{i=1}^\infty y_i$ exists. By Proposition 4.1.7, T is continuous and hence $\lim_{n\to\infty} z_n$ exists and is equal to $z := x - T(\sum_{i=1}^\infty y_i)$. Let $n \geq 1$. Then $z_m \in \overline{TC_n+D}$ for all $m \geq n$ and hence $z \in \overline{TC_n+D}$.
Then $z \in \bigcap_{n\geq 1}\overline{TC_n+D} = D$, for D is closed and $\lim_{n\to\infty} TC_n = 0$.
Then $x = \sum_{i=1}^\infty Ty_i + z = T(\sum_{i=1}^\infty y_i) + z \in TC+D$.
We obtain that $\lambda\overline{TC+D} \subset TC+D$. This is true for all $\lambda \in B_K^-$ and hence $\overline{TC+D} \subset (TC+D)^e$. The latter implies that $(TC+D)^e$ is closed. □

5.4.7 Corollary *Let (A, τ) and (B, σ) be Hausdorff locally convex B_K-modules. Let C be a complete metrizable locally compactoid submodule of A. Then:*

(i) *If $T : A \to B$ is a continuous homomorphism, then $(TC)^e$ is closed.*

(ii) *If D is a closed submodule of A, then $(C + D)^e$ is closed.*

Proof: (i) This is Theorem 5.4.6 with $D = \{0\}$.
(ii) This is Theorem 5.4.6 with $(B, \sigma) = (A, \tau)$ and T is the identity map. □

5.4.8 Theorem *Let (A, τ) be a complete Hausdorff locally compactoid B_K-module. If K is not spherically complete we assume that (A, τ) is metrizable. Let T be a continuous and injective homomorphism from (A, τ) to a Hausdorff locally convex B_K-module (B, σ).*
Let $(x_\alpha)_{\alpha \in I}$ be a net in A such that $Tx_\alpha \to 0$ in B. Then $\lambda x_\alpha \to 0$ in A for every $\lambda \in B_K^-$.

Proof: case 1. K is spherically complete.
From Theorem 5.3.14 we see that (A, τ) is c-compact. Let $\lambda \in B_K^-$. Let U be an open submodule of A. By using Theorem 5.3.17 we obtain that there exist an $n \in \mathbb{N}$ and submodules $F_1, \ldots, F_n$ of rank 1 of A such that $A = U + (F_1 + \cdots + F_n)$. Then $TA = TU + (TF_1 + \cdots + TF_n)$.
Now U is open and hence closed in A. From Proposition 5.3.10 we obtain that TU is closed in B. TU is absorbing in TA for U is absorbing in A.
$TF_1, \ldots, TF_n$ are submodules of rank 1 of B and TU absorbs every element of $TF_1 + \cdots + TF_n$. $(Tx_\alpha)_{\alpha \in I}$ is a net in $TU + (TF_1 + \cdots + TF_n)$ and $Tx_\alpha \to 0$.
By using Theorem 5.4.5 we obtain that $T(\lambda x_\alpha) = \lambda Tx_\alpha \in TU$ for large α.
As T is injective we obtain that $\lambda x_\alpha \in U$ for large α.
We see that $\lambda x_\alpha \to 0$.
case 2. K is not spherically complete. Then (A, τ) is metrizable.
Let $\lambda \in B_K^-$. Let $\mu \in B_K^-$ such that $|\mu|^3 > |\lambda|$. Let U be an open submodule of A. By Corollary 5.1.22 there exist submodules $F_1, \ldots, F_n$ of A of rank ≤ 1 such that $\mu A \subset (F_1 + \cdots + F_n) + U \subset A$. Now $TF_1, \ldots, TF_n$ are submodules of rank ≤ 1 of B and $\mu TA \subset (TF_1 + \cdots + TF_n) + TU \subset TA$.
Furthermore, U is open and hence closed and therefore a complete metrizable locally compactoid submodule of A. From Corollary 5.4.7 we obtain that $(TU)^e$ is closed.
Now U is absorbing in A and hence TU is absorbing in TA. Furthermore, $(\mu Tx_\alpha)_{\alpha \in I}$ is a net in $(TU)^e + (TF_1 + \cdots + TF_n)$ and $(TU)^e$ absorbs every element of $TF_1 + \cdots + TF_n$. From Lemma 5.4.5 we then obtain that $\mu^2 Tx_\alpha \in (TU)^e$ for large α. Then $\mu^3 Tx_\alpha \in TU$ for large α.
As $|\lambda| < |\mu|^3$ we also have that $T(\lambda x_\alpha) = \lambda Tx_\alpha \in TU$ for large α. Then $\lambda x_\alpha \in U$ for large α, for T is injective.
We see that $\lambda x_\alpha \to 0$. □

5.4.9 Theorem *Let (A, τ) be a complete Hausdorff locally compactoid B_K-module. If K is not spherically complete we assume that (A, τ) is metrizable. Let (B, σ) be a Hausdorff locally convex B_K-module and let $T : A \to B$ be a continuous surjective homomorphism. Let U be an open submodule of A. Then $\lambda^{-1} TU$ is open in B for every $\lambda \in B_K^-$.*

Proof: Let $\lambda \in B_K^-$. Decompose T in the following way.

$$\begin{tikzcd} A \arrow[r, "T"] \arrow[d, "\pi"'] & B \\ A/\operatorname{Ker} T \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Then φ is bijective and continuous.
Suppose that $\lambda^{-1}TU$ is not open in B. Then it is not a zero neighbourhood, hence there exists a net $(z_\alpha)_{\alpha\in I}$ in B with $z_\alpha \to 0$ and $z_\alpha \notin \lambda^{-1}TU$ for all $\alpha \in I$.
Let $(x_\alpha)_{\alpha\in I}$ be a net in $A/\operatorname{Ker} T$ such that $\varphi(x_\alpha) = z_\alpha \quad (\alpha \in I)$.
If K is spherically complete then (A, τ) is c-compact and hence $A/\operatorname{Ker} T$ provided with the quotient topology is c-compact. That is to say that $A/\operatorname{Ker} T$ is a complete (Hausdorff) locally compactoid.
If K is not spherically complete then (A, τ) is metrizable and hence, by Theorem 3.4.15, normable. $\operatorname{Ker} T$ is a closed submodule of A. By using Proposition 3.2.17 and Theorem 5.1.15 we obtain that $A/\operatorname{Ker} T$ is a complete normable and hence metrizable locally compactoid B_K-module.
Hence, we can apply Theorem 5.4.8 and obtain that $\lambda x_\alpha \to 0$ in $A/\operatorname{Ker} T$.
Now $\pi(U)$ is open in $A/\operatorname{Ker} T$ and hence $\lambda x_\alpha \in \pi(U)$ for large α. Then $\lambda z_\alpha = \lambda\varphi(x_\alpha) = \varphi(\lambda x_\alpha) \in \varphi(\pi(U)) = TU$ and hence $z_\alpha \in \lambda^{-1}TU$ for large α. A contradiction.
Hence $\lambda^{-1}TU$ is open. □

Now we will formulate the previous results in terms of the plus- and minus topologies.

5.4.10 Theorem *Let (A, τ) be a complete Hausdorff locally compactoid B_K-module. If K is not spherically complete we assume that τ is metrizable. Let (B, σ) be a Hausdorff locally convex B_K-module and let $T : A \to B$ be a continuous surjective homomorphism. Then:*

1) *$T : (A, \tau) \to (B, \sigma^+)$ is open and $T : (A, \tau^+) \to (B, \sigma^+)$ is a quotient map.*

2) *If the map T is bijective, then $T : (A, \tau^-) \to (B, \sigma)$ is open and $T : (A, \tau^-) \to (B, \sigma^-)$ and $T : (A, \tau^+) \to (B, \tau^+)$ are homeomorphisms.*

Proof: In this proof we will use Proposition 2.1.20 several times.
1) We first prove that $T : (A, \tau^+) \to (B, \sigma^+)$ is a quotient map. The continuity of $T : (A, \tau^+) \to (B, \sigma^+)$ follows from Proposition 4.4.20. We prove that $T : (A, \tau^+) \to (B, \sigma^+)$ is open. To this end let U be a τ^+-open submodule of A. Let $\lambda \in K$ with $|\lambda| > 1$. Let $\nu \in K$ such that $1 < |\nu| < |\lambda|$. Then νU is τ-open in A and hence $\frac{\lambda}{\nu}T(\nu U)$ is σ-open in B. Now

$$\frac{\lambda}{\nu}T(\nu U) \subset \frac{\lambda}{\nu}(\nu TU) = (\frac{\lambda}{\nu}\nu)TU = \lambda TU$$

and hence λTU is σ-open in B.
We see that TU is σ^+-open.

Hence, $T : (A, \tau^+) \to (B, \sigma^+)$ is a quotient map. That $T : (A, \tau) \to (B, \sigma^+)$ is open follows from the openness of $T : (A, \tau^+) \to (B, \sigma^+)$ and the fact that $\tau \le \tau^+$ (see Proposition 4.4.22).
2)(i) We prove that $T : (A, \tau^-) \to (B, \sigma)$ is open.
Let U be a τ^--open submodule of A. Then there exists a $\lambda \in K$ with $|\lambda| > 1$ and a τ-open submodule V such that $\lambda V \subset U$. Now λTV is σ-open in B and $\lambda TV = T(\lambda V) \subset TU$, thus TU is σ-open in B.
(ii) The continuity of $T : (A, \tau^-) \to (B, \sigma^-)$ follows from Proposition 4.4.4. We prove that $T : (A, \tau^-) \to (B, \sigma^-)$ is open. To this end let U be a τ^--open submodule of A. Then there exists a $\lambda \in K$ with $|\lambda| > 1$ and a τ-open submodule V of A such that $\lambda V \subset U$.
Let $\nu \in K$ such that $1 < |\nu| < |\lambda|$. Then νTV is σ-open (see Theorem 5.4.9) and hence $\lambda TV = \frac{\lambda}{\nu}(\nu TV)$ is σ^--open. Now $\lambda TV = T(\lambda V) \subset TU$ and hence TU is σ^--open.
(iii) From 1) we know that $T : (A, \tau^+) \to (B, \sigma^+)$ is a quotient map. As T is bijective we obtain that $T : (A, \tau^+) \to (B, \sigma^+)$ is a homeomorphism. □

5.4.11 Remark If the map T in Theorem 5.4.10 is not bijective, then $T : (A, \tau^-) \to (B, \sigma)$ need not be open.
For example, let $r \in (0,1)$. Let $\mathrm{B}(0,r) = \{\lambda \in B_K \mid |\lambda| \le r\}$. Let $A = B_K$ and $B = B_K/\mathrm{B}(0,r)$. Let the norm $\| \ \|$ on B be defined by

$$\|\lambda + \mathrm{B}(0,r)\| = (|\lambda| - r) \vee 0 \quad (\lambda \in B_K).$$

Then $T : (A, | \ |) \to (B, \| \ \|)$ defined by

$$T(\lambda) = \lambda + \mathrm{B}(0,r) \quad (\lambda \in A)$$

is a continuous homomorphism.
In Remark 4.4.31 we have seen that $| \ |^- = | \ |$, but $T : (A, | \ |) \to (B, \| \ \|)$ is not open for $\mathrm{B}(0, \frac{1}{2}r)$ is open in A and $T(\mathrm{B}(0, \frac{1}{2}r)) = \{0\}$, which is not open in the $\| \ \|$-topology.

In Theorem 5.4.10 we cannot substitute τ^+ by τ^{ind} as we see in the following example.

5.4.12 Example Let $B = B_K/\mathrm{B}(0,r)$ and let $\| \ \|$ be as in Remark 5.4.11. Then the $\| \ \|^{\mathrm{ind}}$-topology equals the $\| \ \|$-topology (see Example 4.4.44).
Now $\mathrm{id}_B : (B, d) \to (B, \| \ \|)$ is continuous but not open.
Thus $\mathrm{id}_B : (B, d) \to (B, \| \ \|^{\mathrm{ind}})$ is also not open.

For a compact Hausdorff space neither a strictly weaker Hausdorff topology, nor a strictly stronger compact topology exists. In the next two corollaries we obtain the version of this fundamental compactness property for locally convex B_K-modules.

5.4.13 Corollary *Let (A, τ) be a complete locally compactoid B_K-module. If K is not spherically complete we assume that τ is metrizable. Let σ be a locally convex Hausdorff topology on A which is weaker than τ. Then $\tau^- \le \sigma$.*

Hence, if A contains no simple submodules, then τ^- is the weakest Hausdorff topology on A that is weaker than τ.

Proof: The map $\mathrm{id}_A : (A, \tau) \to (A, \sigma)$ is continuous and from Theorem 5.4.10 we obtain that $\mathrm{id}_A : (A, \tau^-) \to (A, \sigma)$ is open.
Thus $\tau^- \leq \sigma$. □

5.4.14 Corollary

(i) *Let K be spherically complete. Let (A, τ) be a c-compact module. Let σ be a locally convex topology on A that is stronger than τ, such that (A, σ) is also c-compact. Then $\sigma \leq \tau^+$.*

(ii) *Let (A, τ) be a complete metrizable locally compactoid B_K-module. Let σ be a locally convex topology on A that is stronger than τ such that (A, σ) is a complete metrizable locally compactoid B_K-module. Then $\sigma \leq \tau^+$.*

Proof: The map $\mathrm{id}_A : (A, \sigma) \to (A, \tau)$ is continuous and from Theorem 5.4.10 we obtain that $\mathrm{id}_A : (A, \sigma) \to (A, \tau^+)$ is open. Hence, $\sigma \leq \tau^+$. □

5.4.15 Remark In general it is not true that if K is spherically complete and (A, τ) is a c-compact B_K-module, then (A, τ^+) is also c-compact as we have already seen in Remark 5.3.20. (See also Theorem 5.3.21.)
It is also not true that if (A, τ) is a complete metrizable locally compactoid B_K-module, then (A, τ^+) is also metrizable. In fact, in Corollary 4.4.54 we have seen that the plus topology of the product topology on $B_K^{\mathbb{N}}$ is not metrizable, whereas the product topology itself is metrizable.

The Discrete Case

If the valuation on K is discrete (then in particular K is spherically complete), we can prove that a continuous homomorphism between two c-compact B_K-modules is open. The proof follows more or less the same way as in the case K is dense. But it is much easier.

From now on we assume the valuation on K to be discrete.

5.4.16 Lemma *Let (A, τ) be a locally convex B_K-module. Let B be a closed submodule of A and let F be a submodule of rank 1 of A. Let $C \subset K$ be absolutely convex and let $\varphi : C \to F$ be a surjective homomorphism. Let $(b_\alpha)_{\alpha \in I}$ be a net in B and let $(c_\alpha)_{\alpha \in I}$ be a net in C such that $x_\alpha := b_\alpha + \varphi(c_\alpha) \xrightarrow{\tau} 0$. Then $c_\alpha \to 0$ or $x_\alpha \in B$ for large α. The latter occurs if B absorbs all elements of F.*

The proof is a slight modification of that of Lemma 5.4.1. Here we have that $\limsup_{\beta \in I} |c_\beta|$ belongs to $\{|c_\beta| \mid \beta \in I\}$, which gives us this stronger result.

From this lemma it is not hard to prove the following (compare Lemma 5.4.3).

5.4.17 Lemma *Let (A, τ) be a locally convex B_K-module. Let B be a closed submodule of A and let F be a rank 1 submodule of A. Then $B + F$ is closed.*

The next lemma is obtained by induction (compare Lemma 5.4.4).

5.4.18 Lemma *Let (A, τ) be a locally convex B_K-module and let B be a closed submodule of A. Let $n \in \mathbb{N}$ and let $F_1, \dots, F_n$ be rank 1 submodules of A. Then $B + F_1 + \cdots + F_n$ is closed.*

Now it is not hard to prove the following (compare Lemma 5.4.5).

5.4.19 Lemma *Let (A, τ) be a locally convex B_K-module and let B be a closed submodule of A. Let $F_1, \dots, F_n$ be submodules of rank 1 of A, such that B absorbs every element of $F_1 + \cdots + F_n$. If $(x_\alpha)_{\alpha \in I}$ is a net in $B + (F_1 + \cdots + F_n)$, with $x_\alpha \to 0$, then $x_\alpha \in B$ for large α.*

From here we obtain the next theorem (compare Theorem 5.4.8).

5.4.20 Theorem *Let (A, τ) be a c-compact B_K-module. Let T be a continuous and injective homomorphism from (A, τ) to a Hausdorff locally convex B_K-module (B, σ). Let $(x_\alpha)_{\alpha \in I}$ be a net in A such that $Tx_\alpha \to 0$ in B. Then $x_\alpha \to 0$ in A.*

Finally, we can state:

5.4.21 Theorem *Let (A, τ) be a c-compact B_K-module. Let (B, σ) be a Hausdorff locally convex B_K-module and let $T : A \longrightarrow B$ be a surjective continuous homomorphism. Then ((B, σ) is c-compact and) T is open.*

The proof of this theorem is in the same spirit as that of Theorem 5.4.9.

5.4.22 Corollary *Let (A, τ) be a c-compact B_K-module. Let φ be a bijective continuous homomorphism from (A, τ) to a locally convex Hausdorff B_K-module (B, σ). Then φ is a homeomorphism.*

5.4.23 Corollary *Let (A, τ) be a Hausdorff c-compact B_K-module. Then there does not exist a Hausdorff locally convex topology on A that is strictly weaker than τ. Moreover, there does not exist a locally convex topology σ, strictly stronger than τ, such that (A, σ) is c-compact*

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Index of Symbols

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Summary

A valuation $|\ |$ on a field K is called non-archimedean if it satisfies the strong triangle inequality:

$$|\lambda + \mu| \leq \max(|\lambda|, |\mu|) \quad (\lambda, \mu \in K).$$

It is a crucial point that the unit ball $B_K := \{\lambda \in K \mid |\lambda| \leq 1\}$ is a ring; a so-called valuation ring.

In this thesis we introduce a basic theory for locally convex B_K-modules. Most of the theory is developed for the use of the main Chapter 5. We point out that therefore the basic theory is far from complete.

Chapter 5 is about (locally) compact-like B_K-modules.We there introduce the notions *of finite type, local compactoid, (pure) compactoid* and *c-compact*. These are generalizations of the notions carrying the same name in K-vector space theory.

In Chapter 2 we treat the algebraic theory about B_K-modules, needed in the rest of the thesis. This theory can be found in several books about modules. Here we discuss the very simple notions *torsion, divisibility, homomorphisms, submodules* and *quotients*. Also notions like *finitely generated B_K-modules, B_K-modules of finite rank* and *linearly compact B_K-modules* are introduced. We prove, among other things, that for spherically complete K a finitely generated B_K-module is a direct sum of cyclic B_K-modules, see Corollary 2.2.21.
In the final section of this chapter we introduce the notion *edge complete* (Definition 2.4.7), inspired by the notion of an edged absolutely convex set. This algebraic notion indicates if a module has 'edges '. It is an interesting concept, different from the 'classical' one and deserves more research. However, it plays no further role in the setup of this thesis.

In Chapter 3 we start with *locally convex B_K-modules*. This is a module, provided with a topology, for which the addition and the scalar multiplication are continuous. Furthermore, there exists a base of zero neighborhoods consisting of submodules. We discuss here norms and seminorms. The requirements for a (semi)norm on a B_K-module are (and have to be) more relaxed compared to those for a (semi)norm on a K-vector space. The equivalence of two (semi)norms p and q therefore does not imply that there exist constants c_1 and c_2 such that $c_1 p \leq q \leq c_2 p$. Here the equivalence can be described by monotone functions (see Theorem 3.3.19 en Theorem 3.3.20).
Like in K-vector space theory, a locally convex topology on a B_K-module can be described by continuous seminorms. We discuss various definitions for a base of seminorms for such a locally convex topology.

A major part of Chapter 4 consists of tools for the theory on (local) compact-like B_K-modules. Here we prove that on a torsion free B_K-module of finite

rank there exist exactly one locally convex Hausdorff topology, see Proposition 4.1.4. We also introduce B_K*-modules of countable type* and prove that a submodule of such a module is also of countable type.
In this chapter we also discuss the notion of boundedness. Boundedness of a subset X of a locally convex B_K-module does not imply that every continuous seminorm on X is bounded, see Remark 4.3.4.
Also the plus and minus topology are introduced here. These topologies do not occur in locally convex K-vector space theory and play a basic role in Chapter 5.
Finally we pay attention to the *dual* A of a B_K-module. The concept that we propose here is a generalization of the notion dual space for locally convex K-vector spaces, but it is neither Hom (A, K) nor Hom (A, B_K), see Proposition 4.5.12 and Theorem 4.5.13. We prove, among other things, the Hahn-Banach Theorem, see Theorem 4.5.18).

In Chapter 5 we treat, as mentioned before, the (locally) compact-like B_K-modules. We present a complete connection between the notions (local) compactoids, of finite type and c-compactness, see Theorem 5.1.16, Proposition 5.2.3 and Theorem 5.3.14. Among permanence properties, see e.g. Proposition 5.2.4, we prove that a B_K-module of finite type is embeddable in a product of normed modules of finite rank, see Theorem 5.1.11 and that a compactoid B_K-module is embeddable in a product of finitely generated discrete torsion modules, see Theorem 5.2.5. Another interesting thing that we prove is that a locally convex B_K-module on which every continuous seminorm is bounded, must be a pure compactoid B_K-module, see Theorem 5.2.21.
We also prove that a continuous homomorphism φ between two complete metrizable local compactoids A and B need not be open, but, if we provide B with the plus topology, φ is open, see Theorem 5.4.10.
Finally, we show that on a complete metrizable locally compactoid B_K-module (A, τ) each Hausdorff locally convex topology on A, that is weaker than τ, must be stronger than τ^-. Moreover, if σ is a locally convex topology on A, stronger that τ, such that (A, σ) is a complete metrizable locally compactoid, then σ is weaker than τ^+, see Corollary 5.4.13 and Corollary 5.4.14.

Samenvatting

Een waardering $|\ |$ op een lichaam K heet niet-archimedisch als $|\ |$ voldoet aan de sterke driehoeksongelijkheid:

$$|\lambda + \mu| \le \max(|\lambda|, |\mu|) \quad (\lambda, \mu \in K).$$

Een belangrijk gegeven is dat de eenheisdbol $\{\lambda \in K \mid |\lambda| \le 1\}$ een ring; een zogenaamde valuatie ring is. Deze ring geven we aan met B_K.

In dit proefschrift ontwikkelen we een fundamentele theorie voor lokaal convexe B_K-modulen. Het grootste deel van deze theorie staat in dienst van het belangrijkste deel van dit proefschrift: Hoofdstuk 5 over (lokaal) compact-achtige B_K-modulen. We wijzen er dan ook op dat de fundamentele theorie verre van volledig is.

In Hoofdstuk 5 worden de begrippen *van eindig type, lokaal compactoid, (zuiver) compactoid* en *c-compact* besproken. Dit zijn generalisaties van de gelijknamige begrippen uit de theorie van de K-vectorruimten.

In Hoofdstuk 2 wordt de algebraische theorie behandeld over B_K-modulen voor zover we die nodig hebben in de rest van het proefschrift. Deze theorie is in verschillende boeken over modulen terug te vinden. Hier worden hele simpele begrippen als *torsie, deelbaarheid, homomorfismen, deelmodulen* en *quotiënten* behandeld. Ook begrippen als *eindig voortgebracht B_K-moduul, B_K-moduul van eindige rang* en *lineair compacte B_K-modulen* worden besproken. Bewezen wordt, onder andere, dat voor sferisch volledige K een eindig voortgebracht B_K-moduul een directe som is van cyclische B_K-modulen, zie Corollary 2.2.21.
In de laatste paragraaf van Hoofdstuk 2 wordt het begrip *edge complete* ingevoerd (Definition 2.4.7), geinspireerd door de gerande absoluut convexe verzamelingen. Het is een algebraisch begrip dat aangeeft of een moduul 'randen' heeft. Edge complete is een interessant begrip, dat verschilt van het 'klassieke', dat verder onderzoek verdient. Het speelt echter geen rol in de rest van het proefschrift.

In Hoofdstuk 3 wordt het begrip *lokaal-convex B_K-moduul* ingevoerd. Dat is een B_K-moduul voorzien van een topologie zodat de optelling en de scalairvermenigvuldiging continu zijn. Bovendien moet er een basis van nulomgevingen zijn, bestaande uit deelmodulen. Ook normen en seminormen worden hier behandeld. De eisen voor een (semi)norm op een B_K-moduul zijn (noodzakelijk) minder zwaar dan die voor een (semi)norm op een K-vectorruimte. De equivalentie van twee (semi)normen p en q impliceert dan ook niet dat er constanten c_1 en c_2 bestaan zodat $c_1 p \le q \le c_2 p$. Hier kan de equivalentie beschreven worden met behulp van monotone functies (zie Theorem 3.3.19 en Theorem 3.3.20).
Net als in K-vectorruimte theorie kan een lokaal-convexe topologie op een B_K-moduul beschreven worden met behulp van continue seminormen. Er

worden verschillende definities voor een basis van seminormen voor zo'n lokaal convexe topologie gegeven.

Hoofstuk 4 is een bijeenraapsel van wat onderwerpen uit de lokaal-convexe B_K-moduul theorie. Een belangrijk gedeelte is gereedschap voor de theorie in Hoofdstuk 5. Zo wordt er bewezen dat op een torsievrij B_K-moduul van eindige rang er slechts één lokaal convexe Hausdorff topologie bestaat (zie Proposition 4.1.4). Ook worden B_K-modulen van aftelbaar type ingevoerd en bewezen wordt dat een deelmoduul van zo'n moduul weer van aftelbaar type is (zie Theorem 4.2.14).
In dit hoofstuk wordt ook het begrip *begrensdheid* ingevoerd. De begrensdheid van een deelverzameling X van een lokaal convex moduul betekent niet dat elke seminorm op X begrensd is (zie Remark 4.3.4).
Ook worden de min en plus topologie hier ingevoerd. Deze topologieën komen niet voor in de theorie van lokaal convexe K-vector ruimten. Ze spelen een rol in Hoofstuk 5.
Tenslotte wordt aandacht besteed aan de *duale* van een B_K-moduul. De definitie van de duale van een B_K-moduul A die we hier voorstellen is een uitbreiding van het begrip *duale* voor lokaal convexe K-vectorruimten, het is echter niet $\mathrm{Hom}(A, K)$ of $\mathrm{Hom}(A, B_K)$, zie Proposition 4.5.12 en Theorem 4.5.13. We bewijzen onder andere de stelling van Hahn-Banach, zie Theorem 4.5.18.

In Hoofdstuk 5 behandelen we, zoals al eerder is vermeld, de (lokaal) compact-achtige B_K-modulen. We geven de verbanden tussen de begrippen (lokaal) compactoid, van eindig type en c-compact, zie Theorem 5.1.16, Proposition 5.2.3 and Theorem 5.3.14. Naast behoudseigenschappen, zie o.a. Proposition 5.2.4, bewijzen we dat een B_K-moduul van eindig type is in te bedden in een product van genormeerde modulen van eindige rang, zie Theorem 5.1.11 en dat een compactoid B_K-moduul is in te bedden ineen product van eindig voortgebrachte discrete torsiemodulen, zie Theorem 5.2.5. Een ander interessant feit dat wordt bewezen is dat een lokaal convex B_K-moduul, waarvoor alle continue seminormen begrensd zijn, een zuiver compactoid is, zie Theorem 5.2.21.
We bewijzen hier ook dat een continu homomorfisme φ tussen twee volledige metrizeerbare lokale compactoiden A en B niet open hoeft te zijn, doch als men B voorziet van de plus topologie dan is φ wel open., zie Theorem 5.4.10.
Tenslotte laten we zien dat op een compleet metrizeerbaar lokaal compactoid (A, τ) elke Hausdorff lokaal convexe topologie, die zwakker is dan τ, sterker is dan τ^-. Bovendien, als σ een lokaal convexe topologie is op A, sterker dan τ, zodat (A, σ) een compleet metrizeerbaar lokaal compactoid is, dan is σ zwakker dan τ^+, zie Corollary 5.4.13 en Corollary 5.4.14.

Curriculum Vitae

Ik ben geboren op 6 maart 1968 in Wageningen. Daar doorliep ik van 1972 tot 1980 de montessori basisschool de Eekmolen en vervolgens, van 1980 tot 1986, het ongedeeld VWO aan het Wagenings Lyceum.
Van 1986 tot 1989 studeerde ik wiskunde aan de Katholieke Universiteit in Nijmegen, met als afstudeerrichting analyse.
Van 1989 tot 1990 verrichtte ik, als assistent in opleiding, onderzoek in de speltheorie onder leiding van Professor S.H. Tijs. In 1990 verruilde ik de speltheorie voor de p-adische functionaal analyse, waarin ik tot eind 1994 onderzoek verrichtte onder leiding van dr W.H. Schikhof. Eerst drie jaar als assistent in opleiding, vervolgens 14 maanden als junior onderzoeker. De resultaten van dit onderzoek zijn terug te vinden in dit proefschrift.
Vanaf september 1995 ben ik werkzaam als docent wiskunde aan de Regionale Scholengemeenschap Wageningen-Rhenen-Kesteren. Daarnaast volg ik de universitaire lerarenopleiding in Nijmegen.

Stellingen behorende bij het proefschrift
Locally Convex Modules over Valuation Rings
van Saskia Oortwijn

1. Er bestaat een oplossingsconcept voor NTU-spelen dat efficiënt, symmetrisch, monotoon en covariant met affiene transformaties is en samenvalt met de τ-waarde voor TU-spelen en met de Kalai-Smorodinsky oplossing voor onderhandelingsproblemen. (Zie P. Borm, H. Keiding, R.P. McLean, S. Oortwijn en S.H. Tijs, *The Compromise Value for NTU-games*, International Journal of Game Theory 21 (1992), 175–189.)

2. Niet elk zuiver NTU-spel met een niet-lege onderhandelingsverzameling is zwak gebalanceerd. (In *Dekpuntstellingen en de Onderhandelingsverzameling*, doctoraalscriptie van Saskia Oortwijn, 1989. Naar een open probleem van Vohra in *An Existence Theorem for a Bargaining Set*, Journal of Mathematical Economics 20 (1991), 19–34.)

3. Een oplossingsconcept voor een onderhandelingsspel dat Pareto optimaal, covariant met affiene transformaties en onafhankelijk van irrelevante alternatieven is, is ook individueel rationeel. Zo'n oplossingsconcept is noodzakelijkerwijs een dictatoriale oplossing of een (niet noodzakelijk symmetrische) Nash-oplossing. (Zie ook J.C. Harsany en R. Selten, *A Generalized Nash Solution for Two-Person Bargaining Games with Incomplete Information*, Management Science 18 (1972), 80–106 en R. de Koster, H.J.M. Peters, S.H. Tijs en P. Wakker, *Risk Sensitivity, Independence of Irrelevant Alternatives and Continuity of Bargaining Solutions*, Mathematical Social Sciences 4 (1983), 295–300.)

4. Normaal-waarschijnlijkheidspapier geeft geen uitsluitsel of een stochast normaal verdeeld is. Het aantal ogen bij een worp met een (zuivere) dobbelsteen wordt als bij benadering normaal verdeeld aangewezen.

5. Als het aantal leerlingen dat onterecht wiskunde B kiest blijft stijgen — ondanks pogingen dit aantal terug te dringen — zit er niets anders op dan het niveau van dit vak te verlagen.

6. Docenten wiskunde op middelbare scholen en universiteiten dienen er zich terdege van bewust te zijn dat, doordat de huidige wiskundige opvoeding van jongeren zeer toepassingsgericht en concreet is, formele aanpakken door hen niet meer, zoals vroeger, voor zoete koek worden geslikt.

7. Het gedogen van het massaal overtreden van de verkeersregels doet vermoeden dat deze regels slechts gehandhaafd worden ter wille van het niveau van het theoretisch gedeelte van het rijexamen.

8. Het *Kijk woordenboek wiskunde* (bladzijde 232) geeft als omtrek van de ellips met halve assen a en b

$$2\pi\sqrt{\frac{a^2+b^2}{2}}\,.$$

Een veel betere benadering van de omtrek is

$$2\pi\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\,,$$

waarbij $p = {}^{\frac{1}{2}\pi}\log 2$; de maximale fout is kleiner dan 4 promille.

9. Het wetenschappelijk klimaat in Nederland valt niet af te lezen aan de waardering die Nederland heeft voor de grote wiskundigen die het zelf heeft voortgebracht.

10. Wiskunde-onderwijs aan heterogene groepen kan het best gegeven worden volgens de montessori-methode. Het is dan ook een goede greep van de minister om in 1998 onder de vlag van de tweede fase het montessori-onderwijs integraal in de bovenbouw van het havo en vwo in te voeren.

11. Het getuigt van arrogantie als men zich over andere volkeren slechts weet uit te drukken in stereotypen.